THE GRAVITATIONAL FIELD OF A RADIATING CYLINDER

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We present here two solutions for the gravitational field of a radiating cylinder. We have shown tnat one of the solutions represents the electromagnetic radiation caused by fluctuation of current in a straight wire while the other relates to the radiation due to incandescence or flourescence of a cylindrical body.

1. Introduction

Vaidya (1951) has obtained the gravitational field of a radiating sphere (star). We have extended his analysis to a radiating charged sphere Krori and Barua 1974). We discuss here the problem of a radiating cylinder. There can be electromagnetic radiation from cylindrical bodies for either of the two reasons: (i) due to fluctuation of current in a straight wire or (ii) due to incandescence or flourescence. We have obtained here two solutions for the gravitational field of a radiating cylinder. We have shown that one of the solutions represents the electromagnetic radiation caused by fluctuation of current in a straight wire while the other relates to the radiation due to incandescence or flourescence of a cylindrical body.

2. The Field Equations and Their Solutions

The field equations are

$$R^{\nu}_{\mu} - \tfrac{1}{2} R \, \delta^{\nu}_{\mu} = -8\pi T^{\nu}_{\mu} \quad \cdots \quad (2\cdot1)$$

where T^{ν}_{μ} is the energy tensor and is given by (Vaidya 1951)

$$T^{\mu\nu} = \rho \, V^{\mu} V^{\nu} \quad \cdots \quad (2\cdot2)$$

ρ being the density of radiation and V^{μ} being $\dfrac{dx^{\mu}}{d\tau}$ with $d\tau = dx^1_0 = dx^4_0$ in the natural coordinate system. Since radiant energy travels along null geodesics

$$V_{\mu} V^{\mu} = 0, \quad (V^{\mu})_{,\nu} V^{\nu} = 0. \quad \cdots \quad (2\cdot3)$$

As the flow is radial V^2 and V^3 become zero and $T^2_2 = 0$,

$$T^3_3 = 0, \quad T^1_1 = \rho \, V^1 V_1, \quad T^4_4 = \rho \, V^4 V_4, \quad T_1{}^4 = \rho \, V_1 V^4 \quad \cdots \quad (2\cdot4)$$

Again $V^{\mu} V_{\mu} = 0$ simplifies to*

$$V^1 = V^4 \text{ with } V_1 = -V_4 \quad (2\cdot5)$$

We use here the Rosen line-element (1954)—

$$ds^2 = e^{2(\alpha-\beta)} (dt^2 - dr^2) - r^2 e^{-2\beta} d\phi^2 - e^{2\beta} dz^2. \quad (2\cdot6)$$

The free-space field equations with the help of equations (2·5) reduce to the following :

(i) $R_1^1 + R_4^4 = 0$ (2·7)

$$\text{or } \ddot{\alpha} - \ddot{\beta} - \alpha^{11} + \beta'' - \beta'^2 + \dot{\beta}^2 + \frac{\beta'}{r} = 0 \quad (2\cdot8)$$

(ii) $R_2^2 = R_3^3 = 0$ (2·9)

$$\text{or } \beta'' + \frac{\beta'}{r} - \ddot{\beta} = 0 \quad (2\cdot10)$$

(iii) $R_4^4 + R_1^4 = 0$ (2·11)

$$\text{or } \ddot{\alpha} - \ddot{\beta} - \alpha'' + \beta'' + 2\dot{\beta}^2 - \frac{\alpha'}{r} + \frac{\beta'}{r} - \frac{\dot{\alpha}}{r} + 2\dot{\beta}\beta' = 0 \quad (2\cdot12)$$

Combining eqn. (2·10) with (2·8) and (2·12) we get

$$\ddot{\alpha} - \alpha'' - \ddot{\beta}' + \ddot{\beta}^2 = 0 \quad (2\cdot13)$$

$$\ddot{\alpha} - \alpha'' + 2\ddot{\beta} - \frac{\alpha^2}{r} - \frac{\ddot{\alpha}}{r} + 2\ddot{\beta}\beta' = 0 \quad (2\cdot14)$$

and

$$\beta'' + \frac{\beta'}{r} - \ddot{\beta} = 0 \quad (2\cdot15)$$

Combining eqns. (2·13) and (2·14) we get

$$\left(\dot{\beta} + \beta'\right)^2 = \frac{\alpha' + \ddot{\alpha}}{r}. \quad (2\cdot16)$$

Then, the relevant field equations to be solved are eqns. (2·15) and (2·16).

Let us take the solutions in the form

$$\beta = \beta_0(r) + \beta_1(r, t) \quad (2\cdot17)$$

$$\alpha = \alpha_0(r) + \alpha_1(r, t) + 2A\beta_1 \quad (2\cdot18)$$

where α_0 and β_0 represent the field of the material of the cylindrical body. Solutions of time-independent parts are

$$\alpha_0 = A^2 \log r + B \quad (2\cdot19)$$

$$\beta_0 = A \log r + C \quad (2\cdot20)$$

* We do not consider the $V^1 = -V^4$ case because it does not represent the outward flow of radiation.

where A, B and C are constants of integration. A is associated with mass per unit length of the cylindrical body.

The equations for the time-independent parts reduce to

$$\beta_1'' + \frac{\beta_1'}{r} - \beta_1'' = 0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}21)$$

and

$$\alpha_1' + \alpha_1' = r\,(\beta_1' + \beta_1').^2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}22)$$

Equations (2·21) and (2·22) admit of two simple solutions. We shall find them now.

Solution I

A solution of (2·21) is

$$\beta_1 = M J_0\,(pr)\cos pt \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}23)$$

where M is a constant and J_0 is the Bessel function of the first kind and order zero. We take α_1 in the form

$$\alpha_1 = \gamma_1\,(r)\cos 2pt + \gamma_2\,(r)\sin 2pt + \gamma_3\,(r) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}24)$$

Using eqns. (2·23) and (2·24) in eqn. (2·22), we get the values of γ_1, γ_2, γ_3 as

$$\gamma_1 = \left(C_1\cos 2pr + C_2\sin 2pr\right) + \frac{\sin 2pr}{2p}\int (K' - 2pL)\cos 2\,prdr$$

$$- \frac{\cos 2pr}{2p}\int (K' - 2pL)\sin 2prdr + D_1 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}25)$$

$$\gamma_2 = \left(d_1\cos 2pr + d_2\sin 2pr\right) + \frac{\sin 2pr}{2p}\int\left(L' + 2pK\right)\cos 2prdr$$

$$- \frac{\cos 2pr}{2p}\int\left(L' + 2pK\right)\sin 2prdr + D_2 \qquad \ldots \qquad \ldots \qquad (2{\cdot}26)$$

$$\gamma_3 = \int N dr + D_3 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}27)$$

where

$$\left.\begin{aligned} K\,(r) &= \frac{J^2{}_1 - J_0{}^2}{2}\,M^2p^2\,r \\ L(r) &= J_0\,J_1\,M^2\,p^2\,r \\ N\,(r) &= \frac{J_0{}^2 + J_1{}^2}{2}\,M^2\,p^2\,r \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}28)$$

and ϕC_1, ϕC_2, d_1, d_2 D_1, D_2, D_3 are constants.

Solution II

Solution of (2·21) is

$$\beta_1 = RI_0(pr)\, e^{pt} \qquad \text{(2·29)}$$

where R is a constant and I_0 is the modified Bessel function of first kind and order zero. We take α_1 in the form

$$\alpha_1 = \alpha_2(r)\, e^{pt} \qquad \text{(2·30)}$$

Using equations (2·29) and (2·30) in (2·22)

we get

$$2p\alpha_2 + \alpha_2{}' = R^2 (I_0^2 + I_1^2 + 2\, I_0\, I_1)\, p^2 r \qquad \text{(2·31)}$$

where I_1 is the modified Bessel function of the first kind and order one.

Solution of this equation is

$$\alpha_2 = e^{-2pr} R^2 \int (I_0^2 + I_1^2 + 2\, I_0\, I_1)\, p^2 r e^{2pr}\, dr + D \qquad \text{(2·32)}$$

Therefore

$$\alpha_1 = [e^{-2pr} R^2 \int (I_0^2 + I_1^2 + 2\, I_0 I_1)\, p^2 r e^{2pr}\, dr + D]\, e^{2pt}. \qquad \text{(2·33)}$$

3. Discussion

Solution I

Evidently p in this case represents the periodicity of the fluctuating current which is responsible for α_1 and β_1. The effect of the fluctuating current vanishes when $p = 0$. Then the metric will be governed by α_0 and β_0 due to the material of the wire only. M, C_1, C_2, d_1 and d_2 should be related to A so that in the absence of the wire the radiation field vanishes.

Solution II

In order to understand the significance of p in this case we take p to be negative for definiteness. Then it appears from equations (2·29) and (2·33) that α_1, β_1 decrease with the passage of time and ultimately vanish, leaving only α_0 and β_0 to represent the metric. It is therefore evident that p must be associated with the loss of energy from the cylindrical body by radiation in this case. A physical illustration of this case is represented by a wire which is heated to incandescence and then left alone to lose energy by radiation. R should be related to A so that in the absence of the cylindrical body the radiation field vanishes.

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