INTEGRABILITY THEOREMS FOR DIRICHLET SERIES WITH POSITIVE COEFFICIENTS

by S. M. Mazhar, *Department of Mathematics, Kuwait University, Post Box No. 5969, Kuwait*

and

R. S. Khan, *Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh*

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This note deals with the integrability of Dirichlet series with positive coefficients. A number of results have been obtained which generalize various theorems of Askey and Boas (1968) concerning power series.

§ 1.1. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k \geqslant 0$, $0 \leqslant x < 1$. Let S_n be the partial sum of the series $\sum_{n=0}^{\infty} a_n$ and if $\sum_{n=0}^{\infty} a_n < \infty$, let $r_n = \sum_{k=n}^{\infty} a_k$. Askey and Boas (1968) proved the following integrability theorems concerning power series with positive coefficients.

Theorem A—If $a_k \geqslant 0$, $\sum_{k=0}^{\infty} a_k < \infty$ and $1 < p \leqslant \infty$, then $r_n \in l_n^p$ if, and only if $(1-x)^{-2/p}\,[f(1) - f(x)] \in L^p\ (0, 1)$.

Theorem B—Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \geqslant 0$, $0 \leqslant x < 1$. Then if $0 < p < 1$

$$\int_0^1 f(x)^p \, dx < \infty$$

if, and only if

$$\sum_{n=1}^{\infty} n^{-2} \left(\sum_{k=0}^{n} a_k \right)^p < \infty .$$

Theorem C—Let $0 < p < 1, f(x) = \sum_{k=0}^{\infty} a_k x^k,\ a_k \geqslant 0,\ \triangle^j f(x)$ be defined by the relation $\triangle^j f(x) = f(1) - jf(x) + \binom{j}{2} f(x^2) + \ldots + (-1)^j f(x^j)$ and let $jp > 1$. Then

$$\int_0^1 |\triangle^j f(x)|^p (1-x)^{-2}\, dx < \infty$$

if, and only if

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_k \right)^p < \infty .$$

Theorem D—Let $a_n = \int_0^1 x^n\, d\mu(x)$, where $d\mu(x) \geqslant 0$ and $\int_0^1 d\mu(x)$ exists.

Then for $0 < p < \infty$, $\sum_{n=0}^{\infty} a_n^p < \infty$ if, and only if

$$\int_0^1 (1-x)^{-2} \left(\int_x^1 d\mu(y) \right)^p dx < \infty .$$

Our object in this paper is to obtain certain integrability theorems for Dirichlet series and thus generalize all the results stated above.

§ 1.2. In what follows we shall prove the following theorems. In the sequel we assume that $1 = \lambda_1 < \lambda_2 < \ldots < \lambda_n \rightarrow \infty$ and $f(t) = \sum_{k=0}^{\infty} a_k e^{-\lambda_k t},\quad 0 < t \leqslant \infty$.

Theorem 1—If $1 < p \leqslant \infty$, then $\sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\, r_n{}^p < \infty$ if, and only if

$$e^{-\frac{t}{p}} (1 - p^{-t})^{-\frac{2}{p}} [f(0) - f(t)] \in L^p (0, \infty),$$

where $\left\{ \frac{\lambda_n}{\lambda_{n+1}} \right\}$ is a non-decreasing sequence and $\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n^p} < \infty$.

Thoerem 2—Let $a_k \geqslant 0$ and $0 < p < 1$. Then

$$\int_0^{\infty} e^{-t} (f(t))^p\, dt < \infty$$

if, and only if

$$\sum_{n=1}^{\infty} \frac{(\lambda_{n+1} - \lambda_n)}{\lambda_n \lambda_{n+1}} \left(\sum_{k=0}^{n} a_k \right)^p < \infty,$$

where $\dfrac{\lambda_{nj}}{\lambda} \sim \lambda_j$, $n \to \infty, j=1,2,..$ and $\{\lambda_{n+1} - \lambda_n\}$ is a non-decreasing sequence.

Theorem 3—Let $0 < p < 1$ and $a_k \geqslant 0$. Then $\sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\, \gamma_n^p < \infty$ if, and only if

$$\int_0^\infty e^{-t} (1-e^{-t})^{-2} \mid \triangle^j f(t) \mid^p dt < \infty$$

where $r_n = \sum_{k=n}^{\infty} a_k$

(#) $\{\lambda_{n+1} - \lambda_n\}$ is a non-increasing sequence,

$$(\#\#) \sum_{\gamma=\nu-1}^{\infty} \frac{2^\gamma}{(\lambda_2^\gamma)^{jp}} = 0 \left(\frac{2^\nu}{(\lambda_2^\nu)^{jp}} \right)$$

and $\triangle^j f(t) = f(0) - jf(t) + \binom{j}{2} f(2t) - .. + (-1)^j f(jt)$, j being a positive integer such that $jp > 1$.

Theorem 4—Let $a_n = \int_0^1 x^{\lambda_n} d\mu(x)$, where $d\mu(x) \geqslant 0$ and $\int_0^1 d\mu(x)$ exists.

Then for $0 < p < \infty$, $\sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\, a_n^p < \infty$,

if, and only if

$$\int_0^1 (1-x)^{-2} \left[\int_x^1 d\mu(y) \right]^p dx < \infty$$

where λ_n satisfies the conditions (#) and (##) of Theorem 3.

§ 2.1 We require the following lemmas for the proof of our theorem:

Lemma 1 (Hardy *et al.* 1952) — If $p > 1$, $a_k \geqslant 0$, $\mu_n > 0$ and $\Lambda_n = \sum_{k=1}^{n} \mu_k$

$$A_n = \sum_{k=1}^{n} a_k \mu_k,$$

then

$$\sum_{n=1}^{\infty} \mu_n \left(\frac{A_n}{\Lambda_n}\right)^p < \left(\frac{p}{p-1}\right) \sum_{n=1}^{\infty} \mu_n a_n^p$$

unless a_n's are all zero.

Lemma 2 (Hardy *et al.* 1952)—$r y^{r-1}(x-y) \leqslant x^r - y^r \leqslant r x^{r-1}(x-y)$, $r \geqslant 1$, where x and y are positive.

Lemma 3—We have $\sum_{n-2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n^j} \sum_{k=1}^{n} a_k \lambda_k^j\right)^p$

$$\leqslant K* \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \gamma_n^p$$

where $j \geqslant 1$, $0 < p < \infty$, $1 = \lambda_1 < \lambda_2 < \ldots < \lambda_n \rightarrow \infty$, $\{\lambda_{n+1} - \lambda_n\}$ is a non-increasing sequence, $a_k \geqslant 0$ and

$$\sum_{\gamma=\nu-1}^{\infty} \frac{2^\gamma}{(\lambda_{2^\gamma})^{jp}} = 0\left(\frac{2^\nu}{(\lambda_{2^\nu})^{jp}}\right).$$

Proof of Lemma 3: *Case* (i) : $0 < p \leqslant 1$.

$$\sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n^j} \sum_{k=1}^{n} a_k \lambda_k^j\right)^p$$

$$= \sum_{r=1}^{\infty} \sum_{n=2^\gamma}^{2^{\gamma+1}-1} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n^j} \sum_{k=1} a_k \lambda_k^j\right)^p$$

$$\leqslant \sum_{r=1}^{\infty} 2^\gamma \frac{(\lambda_{2^\gamma+1} - \lambda_{2^\gamma})}{\lambda_{2^r}} \left(\sum_{k=1}^{2^{r+1}-1} a_k \lambda_k^j\right)^p$$

* Where K denotes a positive constant not necessarily the same at each occurrence.

$$\leqslant \sum_{r=1}^{\infty} 2^{\gamma} \frac{(\lambda_{2^{\gamma}+1} - \lambda_{2^{\gamma}}}{(\lambda_{2^{r}})^{jp}} \left(\sum_{\nu=1}^{r+1} (\lambda_{2^{\nu}-1})^{j} \sum_{k=2\nu-1}^{\infty} a_k \right)^p$$

$$\leqslant \sum_{r=1}^{\infty} 2^{\gamma} \frac{(\lambda_{2^{\gamma}+1} - \lambda_{2^{\gamma}})}{(\lambda_{2^{r}})^{jp}} \sum_{\nu=1}^{r+1} (\lambda_{2^{\nu}-1})^{jp} \left(\sum_{k=2^{\nu}-1}^{\infty} a_k \right)^p$$

$$= \sum_{\nu=1}^{\infty} (\lambda_{2^{\nu}-1})^{jp} \left(\sum_{k=2^{\nu}-1}^{\infty} a_k \right)^p \sum_{r=\nu-1}^{\infty} 2^{\gamma} \frac{(\lambda_{2^{\gamma}+1} - \lambda_{2^{\gamma}})}{(\lambda_{2^{\gamma}})^{jp}}$$

$$= O\left\{ \sum_{\nu=1}^{\infty} (\lambda_{2^{\nu}-1})^{jp} \left(\sum_{k=2^{\nu}-1}^{\infty} a_k \right)^p (\lambda_{2^{\nu-1}+1} - \lambda_{2^{\nu-1}}) \frac{2^{\nu}}{(\lambda_{2^{\nu}})^{jp}} \right\}$$

$$= O\left\{ \sum_{\nu=1}^{\infty} 2^{\nu} (\lambda_{2^{\nu-1}+1} - \lambda_{2^{\nu-1}}) \left(\sum_{k=2^{\nu}-1}^{\infty} a_k \right)^p \right\}$$

$$= O\left\{ \sum_{\nu=2}^{\infty} \sum_{n=2^{\nu-2}+1}^{2^{\nu-1}} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=n}^{\infty} a_k \right) \right\}^p.$$

$$= O\left\{ \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=n}^{\infty} a_k \right)^p \right\}$$

$$= O\left\{ \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \gamma_n^p \right\}.$$

Case (ii) : $1 < p < \infty$. Applying Abel's transformation we have

$$\sum_{k=1}^{n} \lambda_k^j (r_k - r_{k+1}) = \sum_{k=1}^{n-1} \triangle \lambda_k^j \sum_{\nu=0}^{k} (r_\nu - r_{\nu+1}) + \lambda_n^j (r_\nu - r_{\nu+1})$$

$$= \sum_{k=1}^{n-1} (\lambda_{k+1}^j - \lambda_k^j) \gamma_{k+1} + \gamma_0 \lambda_1 - \lambda_n^j \gamma_{n+1}$$

and therefore

$$\sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n^j} \sum_{k=1}^{n} \lambda_k^j (\gamma_k - \gamma_{k+1}) \right)^p$$

$$\leqslant K \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n^j} \sum_{k=1}^{n-1} (\lambda_{k+1}^j - \lambda_k^j) \gamma_{k+1} \right)^p$$

$$+ K \sum_{n=2}^{\infty} \frac{(\lambda_{n+1} - \lambda_n)}{\lambda_n^{jp}} + K \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \gamma_{n+1}^{p}$$

$= L_1 + L_2 + L_3$, say.

Applying Lemmas 1 and 2 we have

$$L_1 \leqslant K \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n^j} \sum_{k=1}^{n-1} j\, \lambda_{k+1}^{j-1} (\lambda_{k+1} - \lambda_k)\, \gamma_{k+1} \right)^p$$

$$\leqslant K \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n} \sum_{k=1}^{-1} (\lambda_{k+1} - \lambda_k)\, \gamma_{k+1} \right)^p$$

$$\leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\, \gamma_n^{p}.$$

Obviously $L_3 \leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\, \gamma_n^{p}$.

Also $L_2 \leqslant K \sum_{n=2}^{\infty} \frac{1}{\lambda_n^{jp}} \leqslant K \sum_{k=1}^{\infty} \frac{2^k}{(\lambda_2^k)^{jp}} < \infty$.

This establishes Lemma 3.

3.1 *Proof of Theorem* 1: *Necessity*—We have

$$\int_0^{\infty} e^{-t} (1 - e^{-t})^{-2} (f(0) - f(t))^p \, dt$$

$$= \int_0^1 (1-x)^{-2} \left(\sum_{k=0}^{\infty} a_k (1 - x^{\lambda_k}) \right)^p dx$$

$$= \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda n+1}} (1-x)^{-2} \left(\sum_{k=1}^{n} a_k (1 - x^{\lambda_k}) + \sum_{k=n+1}^{\infty} a_k (1 - x^{\lambda_n}) \right)^p dx$$

$$\leqslant \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1-x)^{-2} \left(\sum_{k=0}^{n} a_k \lambda_k (1-x) + \sum_{k=n+1}^{\infty} a_k \right)^p dx$$

$$\leqslant K \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1-x)^{-2} \left(\frac{1}{\lambda_n} \sum_{k=0}^{n} a_k \lambda_k \right)^p dx$$

$$+ K \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1-x)^{-2} r_n^p \, dx$$

$$= K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n} \sum_{k=0}^{n} a_k \lambda_k \right)^p + K \sum_{1=n}^{\infty} (\lambda_{n+1} - \lambda_n) \gamma_n^p$$

$$\leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\frac{1}{\lambda_n} \sum_{k=0}^{n-1} (\lambda_{k+1} - \lambda_k) \gamma_{k+1} \right)^p$$

$$+ K \sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n^p} + K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \gamma_n^p$$

$$\leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \gamma_n^p + K < \infty .$$

by virtue of Lemma 1 and the hypotheses of the theorem.

Sufficiency—We have

$$\int_0^{\infty} e^{-t} (1 - e^{-t})^{-2} (f(0) - f(t))^p \, dt$$

$$= \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1-x)^{-2} \left(\sum_{k=0}^{\infty} a_k (1-x^{\lambda_k}) \right)^p dx$$

$$\geqslant \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=n}^{\infty} a_k \left\{ 1 - \left(1 - \frac{1}{\lambda_{n+1}} \right)^{\lambda_k} \right\} \right)^p$$

$$\geqslant \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=1}^{\infty} a_k \left\{ 1 - \left(1 - \frac{1}{\lambda_{n+1}} \right)^{\lambda_n} \right\} \right)^p$$

$$\geqslant \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=2}^{\infty} a_k \, (1 - e^{-\frac{\lambda_1}{\lambda_2}}) \right)^p$$

$$= \left(1 - e^{-\frac{1}{\lambda_1}} \right)^p \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \, r_n^p ,$$

in view of the fact that

$$\left(1 - \frac{1}{\lambda_{n+1}} \right)^{\lambda_n} = \left[\left(1 - \frac{1}{\lambda_{n+1}} \right)^{-\lambda_{n+1}} \right]^{-\frac{\lambda_n}{\lambda_{n+1}}} \leqslant \left[\left(1 - \frac{1}{\lambda_{n+1}} \right)^{-\lambda_{n+1}} \right]^{-\frac{\lambda_1}{\lambda_2}}$$

$$\leqslant e^{-\frac{1}{\lambda_2}}$$

since $\left(1 - \dfrac{1}{x} \right)^{-x}$ is a decreasing function of x $(x > 1)$ tending to e as x tends to infinity and $\left\{ \dfrac{\lambda_n}{\lambda_{n+1}} \right\}$ is a non-decreasing sequence.

This completes the proof of Theorem 1.

3.2 *Proof of Theorem* 2: *Necessity*—We have

$$\int_0^{\infty} e^{-t} \, (f(t))^p \, dt = \int_0^1 \left(\sum_{k=0}^{\infty} a_k \, x^{\lambda k} \right)^p dx$$

$$\geqslant \sum^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} \left(\sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{\lambda_n} \right)^{\lambda_k} \right)^p dx$$

$$\geqslant \sum_{n=1}^{\infty} \left(1 - \frac{1}{\lambda_n} \right)^{p\lambda_n} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} \left(\sum_{k=0}^{n} a_k \right)^p dx$$

$$= \sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n \, \lambda_{n+1}} \left(1 - \frac{1}{\lambda_u} \right)^{p\lambda_n} \left(\sum_{k=0}^{n} a_k \right)^p$$

$$\geqslant K \sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n \, \lambda_{n+1}} \left(\sum_{k=0}^{n} a_k \right)^p .$$

Hence by the hypothesis

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n \lambda_{n+1}} \left(\sum_{k=0}^{n} a_k \right)^p < \infty .$$

Sufficiency—We have

$$\int_0^{\infty} e^{-t} (f(t))^p \, dt$$

$$= \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} \left(\sum_{k=0}^{\infty} a_k x^{\lambda_k} \right)^p dx$$

$$\leqslant \sum_{n=2}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left(\sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{\lambda_n}\right)^{\lambda_k} \right)^p$$

$$\leqslant \sum_{n=2}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left(\sum_{k=0}^{n-1} a_k \right)^p$$

$$+ K \sum_{n=2}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left(\sum_{k=n}^{\infty} a_k e^{-\frac{\lambda_k}{\lambda_n}} \right)^p$$

$$\leqslant K + K \sum_{n=2}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \left(\sum_{j=1}^{\infty} e^{-\frac{\lambda_{jn}}{\lambda_1}} \sum_{k=jn}^{(j+1)n-1} a_k \right)^p$$

$$\leqslant K + K \sum_{n=2}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \sum_{j=1}^{\infty} e^{-\frac{p\lambda_{jn}}{\lambda_n}} \left(\sum_{k=jn}^{(j+1)^{n-1}} a_k \right)^p$$

$$\leqslant K + K \sum_{j+1}^{\infty} e^{-p\lambda_j k} \sum_{n=2}^{\infty} \frac{\lambda_{(j+1)n} - \lambda_{(j+1)^{n-1}}}{\lambda_{(j+1)n} \lambda_{(j+1)^{n-1}}}$$

$$\times \frac{\lambda_{(j+1)n} - \lambda_{(j+1)^{n-1}}}{\lambda_n \lambda_{n-1}} \frac{(\lambda_n - \lambda_{n-1})}{(\lambda_{(j+1)n} - \lambda_{(j+1)^{n-1}})} \left(\sum_{k=0}^{(j+1)^{n-1}} a_k \right)^p$$

$$\leqslant K + K \sum_{j=1}^{\infty} e^{-K\lambda_j} \lambda_j^2 \sum_{n=2}^{\infty} \frac{\lambda_{(j+1)n} - \lambda_{(j+1)^{n-1}}}{\lambda_{(j+1)n} \lambda_{(j+1)^{n-1}}} \left(\sum_{k=0}^{(j+1)n-1} a_k \right)^p$$

$$\leqslant K + K \sum_{j=1}^{\infty} e^{-K\lambda_j} \lambda_j^2 \sum_{r=1}^{\infty} \frac{\lambda_{r+1} - \lambda_r}{\lambda_{r+1} \lambda_r} \left(\sum_{k=0}^{r} a_k \right)^p$$

$$\leqslant K + K \sum_{j=1}^{\infty} e^{-K\lambda_j} \lambda_j^2 \leqslant K + K \sum_{j=1}^{\infty} \frac{1}{j^2} \leqslant K$$

by virtue of the hypotheses of Theorem 2.

This proves Theorem 2.

3.3 *Proof of Theorem* 3: *Necessity*—We have

$$\int_0^\infty e^{-t} (1 - e^{-t})^{-2} | \Delta^j f(t) |^p \, dt$$

$$= \int_0^\infty e^{-t} (1 - e^{-t})^{-2} \left(\sum_{k=1}^{\infty} a_k (1 - e^{-\lambda_k t}) \right)^p dt$$

$$= \int_0^1 (1 - x)^{-2} \left(\sum_{k=1}^{\infty} a_k (1 - x^{\lambda_k})^j \right)^p dx$$

$$\leqslant \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1 - x)^{-2} \left(\sum_{k=1}^{n} a_k (1 - x^{\lambda_k})^j \right)^p dx$$

$$+ \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1 - x)^{-1} \left[\sum_{k=n}^{\infty} a_k (1 - x^{\lambda_k})^j \right]^p dx$$

$$\leqslant \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=1}^{n} a_k \left\{ 1 - \left(1 - \frac{1}{\lambda_n} \right)^{\lambda_k} \right\}^j \right)^p$$

$$+ \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \gamma_n^p$$

$$\leqslant \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=1}^{n} a_k \left(\frac{\lambda_k}{\lambda_n} \right)^j \right)^p + K$$

$$\leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) r_n^p + K < \infty .$$

by virtue of Lemmas 2,3 and the hypothesis of the theorem.

Sufficiency—Proceeding as in the 'Necessity' part we have

$$\int_0^\infty e^{-t}\,(1-e^{-t})^{-2}\,|\,\triangle^j f(t)|^p\,dt$$

$$\geqslant \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1-x)^{-2}\left(\sum_{k=n}^{\infty} a_k \left\{1-\left(1-\frac{1}{\lambda_{n+1}}\right)^{\lambda_k}\right\}^j\right)^p dx$$

$$\geqslant \sum_{n=1}^{\infty} (\lambda_{n+1}-\lambda_n)\left(\sum_{k=n}^{\infty} a_k \left\{1-\left(1-\frac{1}{\lambda_{k+1}}\right)^{\lambda_n}\right\}^j\right)^p$$

$$\geqslant \sum_{n=1}^{\infty} (\lambda_{n+1}-\lambda_u)\left(\sum_{k=n}^{\infty} a_k \left(1-e^{-\frac{1}{\lambda_2}}\right)^j\right)^p.$$

Hence $\sum_{n=1}^{\infty} (\lambda_{n+1}-\lambda_n)\,\gamma_n^p < \infty$.

This proves Theorem 3.

3.4. *Proof of Theorem* 4: *Necessity* — $\sum_{n=1}^{\infty} (\lambda_{n+1}-\lambda_n)\,\gamma_n^p$

$$= \sum_{n=1}^{\infty} (\lambda_{n+1}-\lambda_n)\left(\sum_{k=1}^{\infty} \int_{1-\frac{1}{\lambda_k}}^{1-\frac{1}{\lambda_{k+1}}} x^{\lambda_n}\,d\mu(x)\right)^p$$

$$\geqslant \sum_{n=1}^{\infty} (\lambda_{n+1}-\lambda_n)\left(\sum_{k=n}^{\infty} \left(1-\frac{1}{\lambda_k}\right)^{\lambda_n} \int_{1-\frac{1}{\lambda_k}}^{1-\frac{1}{\lambda_{k+1}}} d\mu(x)\right)^p$$

$$\geqslant \sum_{n=1}^{\infty} (\lambda_{n+1}-\lambda_n)\left(1-\frac{1}{\lambda_n}\right)^{p\lambda_n}\left(\int_{1-\frac{1}{\lambda_n}}^{1} d\mu(x)\right)^p$$

$$\geqslant \left(1-\frac{1}{\lambda_2}\right)^{p\lambda_2} \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} (1-x)^{-2}\left(\int_x^1 d\mu(y)\right)^p dx$$

$$= K\int_0^1 (1-x)^{-2}\left(\int_x^1 d\mu(y)\right)^p dx$$

and hence

$$\int_0^1 (1-x)^{-2}\left(\int_x^1 d\mu(y)\right)^p dx < \infty .$$

$$\textit{Sufficiency} - \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\, a_n^p$$

$$= \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\left[\sum_{k=1}^{\infty} \int_{1-\frac{1}{\lambda k}}^{1-\frac{1}{\lambda_{k+1}}} x^{\lambda_n}\, d\mu(x)\right]^p$$

$$\leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_u)\left[\sum_{k=1}^{n} \int_{1-\frac{1}{\lambda_k}}^{1-\frac{1}{\lambda_{k+1}}} \left(1-\frac{1}{\lambda_{k+1}}\right)^{\lambda_n} d\mu(x)\right]^p$$

$$+ K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n\left[\sum_{k=n}^{\infty} \int_{1-\frac{1}{\lambda_k}}^{1-\frac{1}{\lambda_{k+1}}} d\mu(x)\right]^p$$

$$= T_1 + T_2, \text{ say.}$$

$$\text{Now}\quad T_2 \leqslant K \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n)\left[\int_{1-\frac{1}{\lambda_n}}^{1} d\mu(x)\right]^p + K$$

$$\leqslant K \sum_{n=2}^{\infty} (\lambda_n - \lambda_{n-1})\left[\int_{1-\frac{1}{\lambda_n}}^{1} d\mu(y)\right]^p + K$$

$$\leqslant K \sum_{n=1}^{\infty} \int_{1-\frac{1}{\lambda_{n-1}}}^{1-\frac{1}{\lambda_n}} (1-x)^{-2}\left(\int_x^1 d\mu(y)\right)^p dx + K < \infty$$

by virtue of the hypothesis.

In view of the hypothesis

$$\left(1-\frac{1}{\lambda_{k+1}}\right)^{\lambda_n} \leqslant e^{-\frac{1}{\lambda_2}\frac{\lambda_n}{\lambda_k}} \leqslant K\left(\frac{\lambda_k}{\lambda_n}\right)^r,$$

where r is some positive integer. Thus applying Lemma 3

$$T_1 \leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=1}^{n} \int_{1-\frac{1}{\lambda_k}}^{1-\frac{1}{\lambda_{k+1}}} \left(\frac{\lambda_k}{\lambda_n} \right)^r d\mu(x) \right)^p + K$$

$$\leqslant K \sum_{n=2}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\sum_{k=n}^{\infty} \int_{1-\frac{1}{\lambda_k}}^{1-\frac{1}{\lambda_{k+1}}} d\mu(x) \right)^p + K$$

$$\leqslant K \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \left(\int_{1-\frac{1}{\lambda_n}}^{1} d\mu(x) \right)^p + K$$

$$\leqslant K \int_{1}^{1} (1-x)^{-2} \left(\int_{x}^{1} d\mu(y) \right)^p dx + K < \infty,$$

by virtue of the hypothesis of the theorem.

This completes the proof of Theorem 4.

§ 4.1 It may be observed that if we take $\lambda_n = n$ and $e^{-t} = x$, then Theorems 1,2 and 3 reduce to Theorems *A*, *B* and *C* respectively, while Theorem *D* is the case $\lambda_n = n$ of Theorem 4. On the other hand by taking suitable values of λ_n we can get several interesting results. Thus for example if we take $\lambda_n = n^2$, $e^{-t} = x$ in Theorem 2 we have

Corollary—If $a_k \geqslant 0$, $0 < p < 1$ and $f(x) = \sum_{0}^{\infty} a_n x^{n^2}$, $0 \leqslant x < 1$,

then $\int_0^1 (f(x))^p \, dx < \infty$

if, and only if

$$\sum_{n=1}^{\infty} n^{-3} \left(\sum_{g=0}^{u} a_k \right)^p < \infty .$$

References

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