REAL ZEROS OF A RANDOM POLYNOMIAL WITH HYPERBOLIC ELEMENTS

by M. SAMBANDHAM,* *Department of Mathematics, Annamalai University, Annamalainagar, Tamil Nadu* 608101

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Let $\{ a_k \}$ be a sequence of martingale differences with

$$E(a_k) = 0,$$
$$E(T_k^2) \sim A^2 \quad (A^2 > 0, \; n \to \infty),$$
$$E|a_k|^{2+\delta} \leqslant M \quad (\text{for somc } \delta > 0 \text{ and } M < \infty)$$

where $T_k = n^{-1/2} \xi_{kn}$, $h = 1, 2, \ldots, n$ and ξ_{kn} is the sum $a_{(k-1)n+1} + \ldots + a_{kn}$.

We show that the average number of real zeros of the polynonial $\sum_{k=1}^{n} \xi_{kn} \cosh kx$ is asymptotic to $\pi^{-1} \log n$, for large n.

1. Consider the polynomial

$$f(x) = \sum_{k=1}^{n} \xi_{kn} \cosh kx \qquad \ldots \quad \ldots \quad \ldots \qquad (1\cdot1)$$

where $\xi_{kn} = a_{(k-1)n+1} + \ldots + a_{kn}$. Let $\{a_k\}$ be a sequence of random variables with

$$\left.\begin{aligned} &E(a_k) = 0 \\ &E(T_k^2) \sim A^2 \quad (A^2 > 0, n \to \infty) \\ &E|a_k|^{2+\delta} \leqslant M \quad (\text{for some } \delta > 0 \text{ and } M < \infty) \end{aligned}\right\} \ldots \qquad (1\cdot2)$$

where $T_k^2 = n^{-1/2} \quad \xi_{kn}$, for $k = 1, 2, \ldots, n$.

If $\{a_k\}$ is a sequence of martingale differences (*md*) satisfying the conditions (1·2) we want to estimate the asymptotic value of the average number of real zeros of (1·1) for large n.

For the polynomial $\sum_{k=1}^{n} p_k \cosh kx$, Das (1971) estimated the average number of real zeros when the random variables p_i are independent and normally distributed with mean zero and variance one.

* *Present address:* Post Graduate Department of mathematics. Ayya Nadar Janaki Ammal College Sivakasi—626124.

2. Let $M(\alpha, \beta)$ be the average number of real zeros of $f(x)$ when x is in (α, β). We estimate the asymptotic value of $M(-\infty, \infty)$ for large n.

Theorem 2·1—Let $\{a_k\}$ be a sequence of *md* satisfying the assumptions (1·2) and

$$E\left| n^{-1} \sum_{k=(p-1)n+1}^{pn} [E(a_k \mid P_k) - E(a_k^2)] \right| \leqslant B(n) \downarrow 0 \qquad .. \qquad (2\cdot1)$$

when $n \to \infty$ for each $p = 1, 2, \cdots, n$. Then $M(-\infty, \infty)$ of (1·1) is asymptotic to $\pi^{-1} \log n$ for large n. (Here P_k are the past events generated by the random variables $\{a_{k-1}, a_{k-2}, \cdot\cdot\}$)

Proof : According to the assumptions $\{a_k\}$ is a sequence of *md*. In this case if the correlations exist the random variables a_i are uncorrelated. This shows that each ξ_{kn} are uncorrelated. Since each $\xi_{k,n}$ satisfies the conditions (1·2) and (2·1) from Serfling (1968, Theorem 6·1) we find that $\xi_{k,n}$ are asymptotically normal with mean zero and variance n, if $E(a_k^2) = 1$. Hence $f(x)$ is asymptotically normal. Using the result of Cramer and Leadbetter (1967) we get

$$M(0, 1) \backsim \int_0^1 \gamma\sigma^{-1}(1 - \varrho^2)^{1/2} \phi(m/\sigma) \times [(2\phi(\eta) + \eta(2\Phi(\eta) - 1)] \, dx$$

where

$$\sigma^2 = \sigma^2(x) = \operatorname{var} f(x),$$
$$\gamma^2 = \gamma^2(x) = \operatorname{var} f'(x),$$
$$\varrho = \varrho(x), \ \gamma\sigma\varrho = \operatorname{cov}(f(x), f'(x))$$

and $\eta = \eta(x) = [(m' - \gamma \varrho m)/\sigma] [\gamma (1 - \varrho^2)^{1/2}]^{-1}$

where ϕ and Φ are standard normal density and distribution function respectively, $E(f(x)) = m(x)$ and $m'(x)$, its continuous derivatives.

Substituting the corresponding values we get

$$\sigma^2 = n \sum_{k=1}^{n} \cosh^2 kx$$

$$\gamma^2 = n \sum_{k=1}^{n} k^2 \sinh^2 kx$$

$$\gamma\sigma\varrho = n \sum_{k=1}^{n} k \cosh kx \sinh kx$$

$$m(x) = m'(x) = 0$$

$$\phi(o) = (2\pi)^{-1|2}$$

$$\eta = 0.$$

Using these values we find that $M(0, 1)$ is asymptotic to

$$(2\pi)^{-1} \int_0^1 [(AC - B^2)^{1/2}/A] \, dx$$

where

$$A = A(x) = \sum_{k=1}^{n} \cosh kx$$

$$B = B(x) = \sum_{k=1}^{n} k \cosh kx \sinh kx$$

$$C = C(x) = \sum_{k=1}^{n} k^2 \sinh^2 kx.$$

Hence $M(-\infty, \infty) \backsim \pi^{-1} \log n$ for large n. Hence the proof.

The relation (2·1) is relatively mild dependent resriction. It is satisfied if

$$E \mid E(a_k^2 \; P_k) - E(a_k^2) \mid \to 0, \; n \to \infty.$$

This leads to the following

Corollary 2·1—If $\{a_k\}$ is a sequence of *md* for which $E(a_k^2) = \sigma^2 < \infty$, $E \mid a_k \mid^{2+\delta} \leqslant M < \infty$ for some $\delta > 0$ and

$$E \mid E(a_k^2 \mid P_k) - \sigma^2 \mid \to 0, \; n \to \infty,$$

then $M(-\infty, \infty) \backsim \pi^{-1} \log n$ for large n.

PROOF : Same as in the above theorem.

If $\{a_k\}$ is a sequence of *md* with $E(a_k^2) = \sigma^2 < \infty$ then $\{a_k\}$ is weakly stationary. Hence corollary (2·1) relates to weakly stationary sequences. If $\{a_k\}$ is a strictly stationary ergodic sequence of *md* then we get the

Theorem 2·2—If $\{a_k\}$ is a strictly stationary ergodic sequence of *md* with $E(a_k^2) = \sigma^2 < \infty$, then $M(-\infty, \infty) \backsim \pi^{-1} \log n$ for large n.

PROOF : From Serfling (1968, Theorem 6·2) which is a theorem of Ibragimov, we find that if $\{a_k\}$ is a strictly stationary sequence of *md* with $E(a_k^2) = \sigma^2 < \infty$ then ξ_{kn} are asymptotically normal. The rest of the proof follows from Theorem 2·1.

We state the following condition (I) : for any set $B \in M_k^{\infty}$ with probability one

$$\mid P(B \mid M_{-\infty}^{k}) - P(B) \mid \leqslant \phi(k) \downarrow 0 \; (k \to \infty).$$

Corollary 2·2—If $\{a_k\}$ is a sequence of *md* satisfying the assumptions (1·2) and condition (I) then $M(-\infty, \infty) \sim \pi^{-1} \log n$, for large n.

PROOF : If $\{a_k\}$ is a sequence of *md* satisfying the assumptions (1·2) and condition (I) then from Serfling (1968, Corollary 6.1.2.) each ξ_{kn} is asymptotically normal. Hence from Theorem 2.1 we get the proof.

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