ON COMMON FIXED POINTS

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Two sets of sufficieut conditions for the existence of a unique common fixed point of a finite number of selfmappings of a complete metric space have been presented.

1. Throughout this paper X will denote a complete metric space (X, d). A mapping $T : X \rightarrow X$ is called a contraction mapping if these exists a constant k, $0 \leqslant k < 1$ such that

$$d(Tx, Ty) \leqslant k\, d(x, y) \text{ for all } x, y \text{ in } X.$$

The well known Banach contraction theorem states that if T be a contraction selfmapping of X then there is a unique point ξ in X such that $T\xi = \xi$, ξ is called the fixed point of T. Kannan (1968) proved the following:

Theorem A Kannan (1968)—If T be a mapping of X into itself such that

$$d(Tx, Ty) \leqslant k\,[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, $0 < k < \frac{1}{2}$, then there is a unique fixed point of T in X.

Singh (1969) obtained the following generalization of Theorem A.

Theorem B (Singh 1969)—If T be a mapping of X into itself and if T (n is a positive integer)

satisfies

$$d(T^n x, T^n y) \leqslant \alpha\,[d(x, T^n x) + d(y, T^n y)] \text{ for all } x, y \in X,\ 0 < \alpha$$

$< \frac{1}{2}$, then T has a unique fixed point in X.

Srivastava and Gupta (1971) established the following generalization of Theorem B.

Theorem C—(Srivastava and Gupta 1971) Let T be a mapping of X into itself such that

$$d(T^p x, T^q y) \leqslant \alpha\, d(x, T^p x) + \beta\, d(y, T^q y)$$

where $x, y \in X$, p, q are positive integers, $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$, then T has a unique fixed point in X.

More recently we have established the following theorem:

Theorem D (Ray in press)— Let T be a mapping of X into itself such that

$$d(T^p x, T^q y) \leqslant \alpha\, d(x,y) + \beta\,[d(x, T^p x) + d(y, T^q y)] + \gamma\,[d(x, T^q y) + d(y, T^p x)]$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha+2\beta+2\gamma < 1$ $x, y \in X$, p, q, are positive integers, then there is a unique fixed point of T in X.

2. In this section we present a more general result in this direction. Theorems A, B and C become corollaries to our theorem whereas Theorem D becomes a particular case of our theorem.

Theorem 1—If T_1 and T_2 be two selfmaps of X and if

(i) $d(T_1^p T_2^q x, T_1^p T_2^q y)$

$$\leqslant \alpha\, d(x,y) + \beta\, d(x, T_1^p T_2^q x) + \gamma d(y, T_1^p T_2^q y) + \delta\,[d(x, T_1^p T_2^q y) + d(y, T_1^p T_2^q x)],$$

$x, y \in X$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $\alpha + \beta + \gamma + 2\delta < 1$ and p, q are positive integers,

(ii) $T_1 T_2 = T_2 T_1$ for all $x \in X$,

then T_1 and T_2 have a unique common fixed point in X.

Proof : We define a sequence of elements $\{x_n\}$ in X as follows:

Let $x_0 \in X$. be arbitrary. Set $T = T_1^p T_2^q$. We write $x_1 = Tx_0$ and for $n > 1$, $x_n = Tx_{n-1}$. we denote $\dfrac{\alpha + \beta\,\delta}{1 - \gamma - \delta}$ by r. Since $\alpha + \beta + \gamma + 2\delta < 1$, we get $r < 1$.

Now

$$\begin{aligned} &d(x_1, x_2) \\ &= d(Tx_0, Tx_1) \\ &\leqslant \alpha\, d(x_0, x_1) + \beta\, d(xo, Txo) + \gamma\, d(x_1, Tx_1) \\ &+ \delta\,[d(x_0, Tx_1) + d(x_1, Tx_0)] \end{aligned}$$

Therefore $d(x_1, x_2) \leqslant \dfrac{\alpha + \beta + \delta}{1 - \gamma - \delta}\, d(x_0, x_1) = r\, d(xo, x_1)$.

Also

$$d(x_2, x_3) = d(Tx_1, Tx_2) \leqslant \alpha\, d(x_1, x_2) + \beta\, d(x_1, Tx_1) + \gamma\, d(x_2, Tx_2) + \delta\, [(d(x_1, Tx_2) + d(x_2, Tx_1)]$$

i.e. $$d(x_2, x_3) \leqslant \frac{\alpha + \beta + \delta}{1 - \gamma - \delta}\, d(x_1, x_2) \leqslant r^2\, d(x_0, x_1).$$

Similarly, we have in general, $d(x_n, x_{n+1}) \leqslant r^n\, d(x_0, x_1)$.

Hence

$$\begin{aligned} d(x_n, x_{n+m}) &\leqslant d(x_n, x_{n+1}) + \ldots\ldots + d(x_{n+m-1}, x_{n+m}) \\ &\leqslant (r^n + r^{n+1} + \ldots\ldots + r^{n+m-1})\, d(x_0, x_1) \\ &< \frac{r^n}{1 - r}\, d(x_0, x_1) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This proves that $\{x_n\}$ is Cauchy. Since X is complete, so there exists a $\xi \in X$ such that $\lim x_n = \xi$.

At first we show that $T\xi = \xi$.

we have

$$\begin{aligned} & d(\xi, T\xi) \\ & \leqslant d(\xi, x_n) + d(x_n, T\xi) \\ & = d(\xi, x_n) + d(Tx_{n-1}, T\xi) \\ & \leqslant d(\xi, x_n) + \alpha\, d(x_{n-1}, \xi) + \beta\, d(x_{n-1}, Tx_{n-1}) \\ & + \gamma d(\xi, T\xi) + \delta\, [\, d(x_{n-1}, T\xi) + d(\xi, Tx_{n-1})) \\ & \leqslant d(\xi, x_n) + \alpha\, d(x_{n-1}, \xi) + \beta\, d(x_{n-1}, x_n) + \gamma d(\xi, T\xi) \\ & + \delta\, [\, d(x_{n-1}, \xi) + d(\xi, T\xi) + d(\xi, x_n)\,] \end{aligned}$$

Hence

$$(1 - \gamma - \delta)\, d(\xi, T\xi) \leqslant (1 + \delta)\, d(\xi, x_n) + (\alpha + \delta)\, d(x_{n-1}, \xi) + \beta\, d(x_{n-1}, x_n)$$

The expression on the right hand side can be made arbitrarily small by choosing n sufficiently large. Accordingly, $d(\xi, T\xi) = 0$.

we further show that ξ is the unique fixed point of T. Suppose for some $\eta \in X$ we have $\eta = T\eta$ also.

Then $\quad d(\xi, \eta) = d(T\xi, T\eta) \leqslant \alpha\, d(\xi, \eta) + 2\delta\, d(\xi, \eta),$
which gives $\xi = \eta$.

Thus ξ is unique fixed point of T. Now we shall show that $T_1\, \xi = T_2\, \xi = \xi$.

We have

$$T\xi = T_1^p\, T_2^q\, \xi = \xi$$

$$\Rightarrow T_2\,(T_1^p\, T_2^q\, \xi) = T_2\, \xi$$

Or $\quad T_1^p\, T_2^q\,(T_2\, \xi) = T_2\, \xi \quad$ (by using (ii))

Or $\quad T\, T_2\, \xi = T_2\, \xi.$

But ξ is the unique fixed point of T.

Hence $T_2\, \xi = \xi$. Similarly $T_1\, \xi = \xi$.

Remarks : For $T_2 x = x \forall x \in X$, $\beta = \gamma$, we get a particular case of theorem D (namely when $p = q$)

3. In this section we establish another set of sufficient conditions under which two selfmappings defined on X have a unique common fixed point. We prove the following:

Theorem 2—Let T_1 and T_2 be two self mappings of X such that

$$(**) \qquad \begin{aligned} & d(T_1\, T_2 x, T_2 T_1 y) \\ & \leqslant \alpha\, d(x, y) + \beta d(x, T_1\, T_2 x) + \gamma\, d(y, T_2\, T_1 y) \\ & + \delta\, [d(x, T_2\, T_1 y) + d(y, T_1\, T_2 x)] \end{aligned}$$

for all $x, y, \in X$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma + 2\delta < 1$.

Then T_1 and T_2 have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Define a sequence of elements $\{x_n\}$ in X as follows: $x_1 = T_1\, T_2\, x_0$, and, for $n > 1$.

$$x_n = T_1\, T_2 x_{n-1} \qquad \text{when } n \text{ is odd,}$$

$$x_n = T_2\, T_1 x_{n-1} \qquad \text{when } n \text{ is even.}$$

Now

$$\begin{aligned} & d(x_1, x_2) \\ & = d(T_1\, T_2 x_0, T_2\, T_1\, x_1) \\ & \leqslant \alpha\, d(x_0, x_1) + \beta d(x_0, T_1\, T_2 x_0) + \gamma d(x_1, T_2\, T_1\, x_1) \\ & + \delta\, [d(x_0. T_2\, T_1\, x_1) + d(x_1, T_1\, T_2 x_0)] \\ & \leqslant \alpha\, d(x_0, x_1) + \beta d(x_0, x_1) + r d(x_1, x_2) + \delta\, [d(x_0, x_1) + d(x_1, x_1)] \end{aligned}$$

i.e. $$d(x_1, x_2) \leqslant \frac{\alpha+\beta+\delta}{1-\gamma-\delta} d(x_0, x_1).$$

Also

$$\begin{aligned} d(x_2, x_3) &\leqslant d(T_2 T_1 x_1, T_1 T_2 x_2) \\ &\leqslant \alpha d(x_2, x_1) + \beta d(x_2, T_1 T_2 x_2) \\ &\quad + \gamma d(x_1, T_2 T_1 x_1) + \delta [d(x_2, T_2 T_1 x1) + d(x_1, T_1 T_2 x_2)] \end{aligned}$$

i.e. $$d(x_2, x_3) \leqslant \frac{\alpha+\gamma+\delta}{(1-\beta-\delta)} d(x_1, x_2)$$

$$\leqslant \frac{\alpha+\gamma+\delta}{1-\beta-\delta} \quad \frac{\alpha+\beta+\delta}{1-\gamma-\delta} \quad d(x_0, x_1).$$

Put $r_1 = \dfrac{\alpha+\beta+\delta}{1-\gamma-\delta}$ and $r_2 = \dfrac{\alpha+\gamma+\delta}{1-\beta-\delta}$

Since $\alpha+\beta+\gamma+2\delta < 1$, we have $r_1 < 1$ and $r_2 < 1$.

Then in general $d(x_{2n}, x_{2n-1}) \leqslant r_1^n r_2^n d(x_0, x_1)$

and $d(x_{2n+1}, x_{2(n+1)}) \leqslant r_1^{n+1} r_2^n d(x_0 . x_1)$

Hence for $m = 2n$ we have

$$\begin{aligned} & d(x_m, x_{m+p}) \\ & \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots\ldots + d(x_{m+p-1}, x_{m+p}) \\ & \leqslant (r_1^n r_2^n + r_2^n r_1^{n+1} + r_2^{n+1} r_1^{n+1} + \ldots\ldots p \text{ terms}) d(x_0, x_1) \\ & < (r_1^n r_2^n + r_2^n r_1^{n+1} + r_2^{n+1} r_1^{n+1} + \ldots\ldots \text{to infinity}) d(x_0, x_1) \\ & = [r_2^n r_1^n (1 + r_1 r_2 + r_1^2 r_2^2 + \ldots) + r_1^{n+1} r_2^n (1 + r_1 r_2 + \ldots\ldots)] d(x_0, x_1) \\ & = [(r_2^n r_1^n)(1+r_1)/(1-r_1 r_2)] d(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty \text{ i.e. m} \to \infty. \end{aligned}$$

Similarly for $m = 2n+1$

$$\begin{aligned} d(x_m, x_{m+p}) &< (r_2^n r_1^{n+1} + r_2^{n+1} r_1^{n+1} + r_2^{n+1} r_1^{n+2} + r_1^{n+2} r_2^{n+2} + \ldots\ldots) d(x_0, x_1) \\ &= \frac{r_1^{n+1} r_2^n (1+r_2)}{1 - r_1 r_2} d(x_0, x_1) \\ &\to 0 \text{ as } n \to \infty \quad \text{i.e. as } m \to \infty. \end{aligned}$$

Thus the sequence $\{x_n\}$ is Cauchy. But X is complete, so $\exists\, \xi \in X$ such that $\lim x_n = \xi$.

We wish to show that $T_1 T_2 \xi = T_2 T_1 \xi = \xi$.

Consider $d(\xi, T_1 T_2 \xi)$.

Now $d(\xi, T_1 T_2 \xi) \leqslant d(\xi, x_n) + d(x_n, T_1 T_2 \xi)$, where n is taken to be even,

Or
$$\begin{aligned} & d(\xi, T_1 T_2 \xi) \\ & \leqslant d(\xi, x_n) + d(T_1 T_2 \xi, T_2 T_1 x_{n-1}) \\ & \leqslant d(\xi, x_n) + \alpha\, d(\xi, x_{n-1}) + \beta d(\xi, T_1 T_2 \xi) \\ & + \gamma d(x_{n-1}, T_2 T_1 x_{n-1}) + \delta\, [d(\xi, T_2 T_1 x_{n-1}) + d(x_{n-1}, T_1 T_2 \xi)] \\ & \leqslant d(\xi, x_n) + \alpha\, d(\xi, x_{n-1}) + \beta d(\xi, T_1 T_2 \xi) + \gamma d(x_{n-1}, x_n) \\ & + \delta\, d(\xi, x_n) + \delta d(x_{n-1}, \xi) + \delta d(\xi, T_1 T_2 \xi) \end{aligned}$$

Or
$$(1 - \beta - \delta)\, d(\xi, T_1 T_2 \xi) \leqslant d(\xi, x_n) + (\alpha + \delta)\, d(\xi, x_{n-1}) + \gamma d(x_{n-1}, x_n) + \delta d(\xi, x_n).$$

For sufficiently large n, the right-hand side expression is arbitrarily small.

Hence $d(\xi, T_1 T_2 \xi) = 0$ i.e. $T_1 T_2 \xi = \xi$.

Similarly one can show that $T_2 T_1 \xi = \xi$.

Finally we show that ξ is the only element of X such that (***) $T_1 T_2 \xi = \xi = T_2 T_1 \xi$.

Consider
$$\begin{aligned} d(\xi, \eta) &= d(T_1 T_2 \xi, T_2 T_1 \eta) \\ &\leqslant \alpha\, d(\xi, \eta) + \beta d(\xi, T_1 T_2 \xi) + \gamma d(\eta, T_2 T_1 \eta) \\ &+ \delta\, [d(\xi, T_2 T_1 \eta) + d(\eta, T_1 T_2 \xi)] \end{aligned}$$
i.e. $d(\xi, \eta) \leqslant (\alpha + 2\delta)\, d(\xi, \eta)$

which gives
$$\xi = \eta.$$

We claim further that $T_1 \xi = T_2 \xi = \xi$.

For, $T_1 T_2 \xi = \xi = T_2 T_1 \xi \Rightarrow T_1 (T_2 T_1 \xi) = T_1 \xi$

or $T_1 T_2 (T_1 \xi) = T_1 \xi$

But ξ is the only fixed point of $T_1 T_2$.

Hence $T_1 \xi = \xi$. Similarly, $T_2 \xi = \xi$

This completes the proof.

Remark 1: Under the conditions of the above theorem ξ is the only common fixed point of T_1 and T_2. That is to say ξ is the unique common fixed point of T_1 and T_2. For, if possible let ξ^* exist in X such that

$$\begin{aligned} & T_1 \xi^* = T_2 \xi^* = \xi^* \\ & \Rightarrow T_1 T_2 \xi^* = T_2 T_1 \xi^* = \xi^* \\ & \Rightarrow \xi^* = \xi \text{ from (***)} \end{aligned}$$

Remark 2 : If we replace condition (**) by the following condition:

$$d\,(T_1^p\, T_2^q\, x,\ T_2^q\, T_1^p\, y)$$
$$\leqslant \alpha\, d(x, y) + \beta d\,(x,\ T_1^p\, T_2^q\, x) + \gamma d(y,\ T_2^q\, T_1^p\, y)$$
$$+\ \delta\ [d\,(x,\ T_2^q\, T_1^p y) + d(y,\ T_1^p\ T_2^q\, x)]$$

then also the conclusion of theorem 2 holds.

For, Theorem 2 together with Remark 1 gives the existence of a unique common fixed point ξ of T_1^p and T_2^q. Hence $T_1^p\, \xi = T_2^q\, \xi = \xi$.

Now

$$T_1^p\, \xi = \xi$$
$$\Rightarrow T_1(T_1^p\, \xi) = T_1\, \xi$$
$$\Rightarrow T_1^p\ (T_1\, \xi) = T_1\, \xi$$
$$\Rightarrow T_1\, \xi = \xi, \text{ since } \xi \text{ is the unique fixed point of } T_1^p.$$

Similarly, $T_2\, \xi = \xi$

Following Remark 1, it follows immediately that ξ is the only common and unique fixed point of T_1 and T_2

Remark 3: If $\beta=\gamma$, $T_2 x = x \forall x \epsilon x$ in remark 2, then it gives theorem D ($p=q$).

References

Kannan, R. : (1968). Some results on fixed points, *Bull. Calcutta math. Soc.*, **60**, 71-76.

Ray Barada K. (in press). On common fixed points and sequence of mapping

Singh, S.P. (1969). Soma results on fixed points, *Yokohama Math. J.*, **18**, 2, 61-64.

Srivastava, P., ahd Gupta, V. K. (1971). A note on common fixed points. *Yakohama Math.* J., **19**, 2, 91-95.