A GENERALIZED METHOD OF SOLUTION OF THE UNSTEADY COMPRESSIBLE BOUNDARY LAYER EQUATIONS

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(Communicated by B. R. Seth, F.N.A.)

(Received 15 August 1974)

A theory of compressible boundary layer in which a two-dimensional body is accelerating through a compressible fluid at rest at infinity, is developed in the present paper. The flight velocity is of the form $U(x, t) = X(x) N(t)$, which has important application in aerodynamics and initial stages of rocket flight. $N(t)$ is assumed to be square-integrable and $X(x)$ belongs to the class c^{∞}. A general solution of the velocity variable, with a solution for the temperature variable, when Prandtl's number $= 1$ has been presented here. Expressions for the boundary layer characteristics, viz, skin friction, displacement thickness, momentum thickness etc. have been obtained. It also discusses a method to determine the point of separation at different moments of time.

Introduction

In this paper another attempt has been made to generalize, the theory of time-dependent boundary layer theory, which was developed by many in comparatively particular cases. Lighthill began the study of incompressible boundary layer with fluctuating velocity in the main-stream having a mean flow. Subsequently, other types of two-dimensional problems were studied by many: Stuart (1955), Glauert (1966), Watson (1958, 1959), Ghosh (1961), Sarma (1964, 1965), Ghoshal (1966, 70, 72), Yang and Huang (1969). Thermal boundary layer theory in a three dimensional flow has recently been developed by Ghoshal and Ghoshal (1970). Watson (1955) studied the boundary layer development on an infinite plate. He has shown that if the plane moves in an incompressible viscous fluid, the velocity distribution is similar at different instants of time, if the velocity of the plate $U(x, t)$ is of the form $U(t) = At^{a}$, or $U(t) = Ae^{ct}$ where 't' is the time. But in the case of a curved two-dimensional body, even if its velocity is of the above forms, it is not possible to get a similarity solution. If the motion is referred to a set of axes fixed on the body in the usual manner, the velocity at infinity will appear to be of the form $X(x)\, N(t)$. A general method of solution of the boundary layer equations for an incompressible flow past a two-dimensional body is developed by Ghoshal (1971), when the modified time variable $\bar{t}$ is finite (t may be even greater than 1).

Presented in the Advance Level Winter Workshop in Fluid Mechanics, I. I. T., New Delhi, December, 1970.

An attempt has been made by the authors to generalize the problem a step further. A solution of the problem of unsteady boundary layer with a distribution of velocity $U(x, t) = X(x)\ N(t)$ in the main-stream is derived here. In the present paper, which is the first part of the treatment, solutions for the velocity variable with expressions for the related boundary layer characteristics and discussions for the determination of the boundary layer separation at different moments of time are presented. A method to determine the temperature distribution when $Pr = 1$ is also indicated.

Analysis

We consider a two-dimensional body accelerating through a compressible fluid at rest at infinity. The flight velocity is of the form $X(x)$, $N(t)$ where $N(t)$ is defined and square integrable, in the interval $(o, \infty]$ and $x(x)$ belongs to the class c^{∞}. It is assumed that Prandtl boundary layer assumptions are valid.

The leading edge is taken as the origin, x-axis in the direction of the body and y-axis perpendicular to it. Prandtl number pr and $c_p =$ specific heat are assumed to be constant. Under these assumptions the governing equations for compressible, non-isothermal, viscous fluid may be written as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) + \frac{\partial}{\partial y}\left(\rho v\right) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad \ldots \quad (2)$$

$$\frac{\partial p}{\partial y} \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\rho\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{1}{Pr}\cdot\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right) + \frac{\mu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{pc_p}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) \qquad \ldots \quad (4)$$

$$\frac{\mu}{\mu_\infty} = \frac{T}{T_\infty} \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$p = \rho RT \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

Here $Pr =$ Prandtl's number is assumed to be constant. For a flat plate the relation between rectangular coordinate system fixed in the fluid as given in Fig 1(a) and

x^* and y^*, and x and y those with reference to a frame fixed in the plate shown in Fig. 1(b) are $x = x^* + \int_0^{t^*} U dt^*$

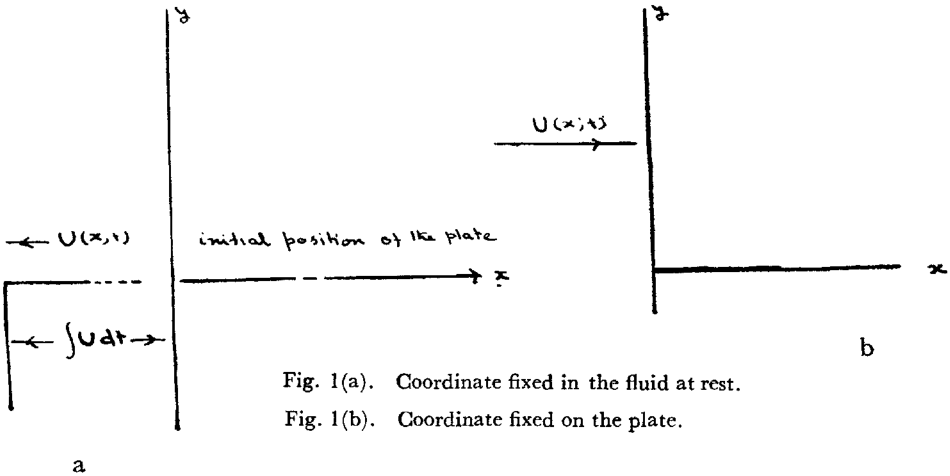

Fig. 1(a). Coordinate fixed in the fluid at rest.

Fig. 1(b). Coordinate fixed on the plate.

$$y = y^*, \quad t = t^*, \quad u = u^* + U, \quad v = v^*$$

The initial and boundary conditions are $u\,(x, y, 0) = U,\ V\,(x,\ y,\ 0) = 0, y = 0;\ u\,(x, 0, t) = v\,(x, 0, t) = 0,\ T\,(x, 0, t) = T_w$ or $\dfrac{\partial T}{\partial y}\,(x, 0, t) = 0,\quad u\,(x,\ \infty\ , t) = U\,(x, t),\quad T(x,\ \infty\ , t) = T_\infty$ (7)

On using a generalization of the Howarth-Dorodnitsin transformation of the type:

$$\rho = \rho_\infty \left(\frac{\delta Y}{\partial y} \right)_{x,\,t} \qquad u = \left(\frac{\partial \psi}{\partial Y} \right)_{x,\,t} \qquad \ldots \qquad (8)$$

and using Y as the independent variable instead of y giving

$$v = -\frac{\rho_\infty}{\rho} \left\{ \left(\frac{\partial \psi}{\partial x} \right)_{Y,t} + v \left(\frac{\partial Y}{\partial x} \right)_{y,t} + \left(\frac{\partial Y}{\partial t} \right)_{x,\,y} \right\} \qquad \ldots \qquad (9)$$

the equations of motion and energy becomes,

$$\frac{\partial^2 \psi}{\partial Y \partial t} + \frac{\partial \psi}{\partial Y} \cdot \frac{\partial^2 \psi}{\partial x \partial Y} - \frac{\partial^2 \psi}{\partial Y^2} \cdot \frac{\partial \psi}{\partial x}\ Ut + UUx + \nu_\infty$$

$$\frac{p_1}{p_\infty}\ \frac{\partial}{\partial y} \left(\frac{\mu T_\infty}{\mu_\infty T}\ \frac{\partial^2 \psi}{\partial Y^2} \right)$$

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial Y} \cdot \frac{\partial T}{\partial x} = \frac{1}{P_r}\ \frac{\nu_\infty p_1}{p_\infty}\ \frac{\partial}{\partial Y} \left(\frac{\mu T_\infty}{\mu_\infty T}\ \frac{\partial T}{\partial Y} \right) + \frac{p_1\, \mu T_\infty}{c_p\, \mu_\infty T \rho_\infty} \left(\frac{\partial^2 \psi}{\partial Y^2} \right)^2$$

$$+ \frac{\partial \psi}{\partial x} \cdot \frac{\partial T}{\partial \bar{Y}} + \frac{T}{c_p \rho_1 T_1} \left(\frac{\partial p_1}{\partial t} + \frac{\partial \psi}{\partial \bar{Y}} \frac{\partial p_1}{\partial x} \right)$$

1-denotes, conditions just outside the boundary layer.

Using (5), we obtain,

$$\psi_{\bar{Y}t} + \psi_{\bar{Y}} \psi_{x\bar{Y}} - \psi_x \psi_{\bar{Y}\bar{Y}} = U_t + UU_x + \bar{\nu} \psi_{\bar{Y}\bar{Y}\bar{Y}} \quad \cdots \quad (10)$$

$$T_t + \psi_{\bar{Y}} T_x - \psi_x T_{\bar{Y}} - \frac{T}{c_p \rho_1 T_1} \left(\frac{\partial p_1}{\partial t} + \psi_{\bar{Y}} \frac{\partial p_1}{\partial_x} \right) = \frac{\bar{\nu}}{P_r} T_{\bar{Y}\bar{Y}} + \frac{\bar{\nu}}{c_p} \left(\frac{\partial^2 \psi}{\partial_Y^2} \right)^2 \quad \cdots \quad (11)$$

where $\bar{\nu} = \nu_\infty \dfrac{p_1(x, t).}{p_\infty}$

Solution of velocity variable

The boundary conditions for ψ are :

$$\left.\begin{aligned} \psi_x &= 0 \quad \text{when } y = 0, \text{ and } t = 0 \\ \psi_x &= \psi_y = 0 \quad \text{when } y = 0 \text{ and } t > 0 \end{aligned}\right\} \quad \cdots \quad (12)$$

Let $c^2 = \int_0^\infty N^2(t)\, dt \qquad \psi_{\bar{Y}} \to U(x, t) \quad \text{as } y \to \infty$

$$\psi = c\bar{\psi}, \ t = t_1/c. \quad \bar{U} = \frac{X(x)\, N(t_1)}{c}, \ \bar{\bar{\nu}} = \frac{\bar{\nu}}{c} = \frac{\nu_\infty p_1(x, t)}{c p_\infty}$$

so that the equation for ψ takes the form

$$\bar{\psi}_{\bar{Y}t} + \bar{\psi}_{\bar{Y}} \psi_{xY} - \bar{\psi}_x \bar{\psi}_{\bar{Y}\bar{Y}} = \bar{U}_t + \bar{U}\bar{U}_x + \bar{\bar{\nu}} \bar{\psi}_{\bar{Y}\bar{Y}Y} \quad \cdots \quad (13)$$

Let us introduce the following independent variables:

$$\left.\begin{aligned} \bar{t} &= \int_0^{t_1} \frac{N^2(t_1)\, dt_1}{c^2}, \ x \\ \text{and } \eta &= \frac{N(t_1)\, \bar{Y}}{k \sqrt{\bar{\nu}_0} \sqrt{3\bar{t}}}, \ \bar{\nu}_0 = \frac{\nu_\infty}{c} \end{aligned}\right\} \quad \cdots \quad (A)$$

for every $x > 0, \ y \geqslant 0, \ t > 0$

Also let us assume $\bar{\psi} = F(x, \eta, \bar{t}) \sqrt{\bar{\nu}_0} \sqrt{3\bar{t}}\, X(x) \quad \cdots$

so that $\bar{u} = \dfrac{\partial \bar{\psi}}{\partial \bar{y}} = \dfrac{u}{c} = \dfrac{\mathcal{N}(t_1)}{c} X(x)\, F\eta\,(x, \eta, \bar{t})$

and the equation becomes

$$\frac{p_1(x, t)}{p_\infty} F\eta\eta\eta + A(\bar{t})\,[1 - F\eta - \eta\, F\eta\eta] + \frac{3}{2}\,\eta\, F\eta\eta - 3\bar{t}\, F_{\eta\bar{t}}$$

$$+ B(\bar{t})\,[X'\,(1 - F\eta^2 + FF\eta\eta) + X\,(Fx\, F\eta\eta - F\eta Fx\eta)] = 0 \qquad (14)$$

where $A(\bar{t}) = 3\bar{t}c^2 \dfrac{\mathcal{N}^1(t_1)}{\mathcal{N}^3(t_1)}$ and $B(\bar{t}) = \dfrac{3\bar{t}c}{\mathcal{N}(t_1)}$ .. .. .. (B)

The boundary conditions are : $F(x, \eta, 0) = 1$

$$F\eta\,(x, 0, \bar{t}) = 0 \qquad F(x, 0, \bar{t}) = 0 \qquad .. \quad .. \quad .. \qquad (15)$$

$$at_{\eta \to \infty}\, F\eta\,(x, \eta, \bar{t}) \to 1.$$

We assume that $X(x)$ possesses derivatives of all orders, so that we seek a solution of the form

$$F(x, \eta, \bar{t}) = F_0(\eta, \bar{t}) + X F_1(\eta, \bar{t}) + X^2 F_2(\eta, \bar{t}) + X'(x)\, F_{12}(\eta, \bar{t}) \qquad (16)$$

The convergence of the series will be discussed in Appendix I.

$\dfrac{p_1(x, t)}{p_\infty}$ may be expressed as follows :

$$\frac{p_1(x, t)}{p_\infty}\, a_0(t) + a_1(t)\, X + a_2(t)\, X' + \qquad .. \quad .. \quad ..$$

Substituting from (1 6) in (14) and equating to zero the coefficients of $X(x)$ and its derivatives and their products etc. we obtain:

$$a_0 F_{0\eta\eta\eta} + A(\bar{t})\,\{1 - F_{0\eta} - \eta\, F_{0\eta\eta}\} + {}^3/_2\, \eta\, F_{0\eta\eta} - 3\bar{t}\, F_{0\eta\bar{t}} = 0 \qquad .. \qquad (17)$$

$$a_0 F_{1\eta\eta\eta} + a_1 F_{0\eta\eta\eta} + A(\bar{t})\,\{- F_{1\eta} - \eta\, F_{1\eta\eta}\} + {}^3/_2\, \eta\, F_{1\eta\eta} - 3\bar{t}\, F_{1\eta\bar{t}} = 0 \qquad (18)$$

$$a_0 F_{2\eta\eta} + a_1 F_{1\eta\eta\eta} + A(\bar{t})\,\{- F_{2\eta} - \eta\, F_{2\eta\eta}\} + {}^3/_2\, \eta\, F_{2\eta\eta} - 3\bar{t}\, F_{2\eta\bar{t}} = 0 \qquad (19)$$

$$a_0 F_{12\eta\eta\eta} + a_2 F_{0\eta\eta\eta} + A(\bar{t})\,\{- F_{12\eta} - \eta\, F_{12\eta\eta}\} + {}^3/_2\, \eta\, F_{12\eta\eta} - 3\bar{t}\, F_{12\eta\bar{t}}$$

$$+ B(\bar{t})\,[1 - F_{0\eta}{}^2 + F_0 F_{0\eta\eta}] = 0 \qquad .. \quad .. \quad .. \qquad (20)$$

Boundary conditions are

$$\left.\begin{array}{llll} \text{(i)} & F_{0\eta} = F_{1\eta} = F_{2\eta} = \ldots\ldots = 0 & \text{at } \bar{t} = 0 & 1\,\eta = 0 \\ \text{(ii)} & F_{0\eta} = F_{1\eta} = F_{2\eta} = \ldots\ldots = 0 & \text{at } \eta = 0 & \bar{t} > 0 \\ & F_0 = F_1 = F_2 = \ldots\ldots = 0 & \text{at } \eta = 0 & \bar{t} > 0 \\ \text{(iii)} & F_{0\eta} \to 1,\ F_{1\eta} \to 0,\ F_{2\eta} \to 0 \text{ etc. } y \to \infty & & \bar{t} > 0 \end{array}\right\} \quad \ldots \quad (21)$$

Now $A\ (\bar{t})$ may be expressed in the form

$$A\ (\bar{t}) = 3\bar{t}\ \frac{d}{dt}\ ln\ \frac{N}{N_0} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (22)$$

where N_0 is a certain constant, so that N/N_0 is dimensionless.

Now, $N\ (t_1) \neq 0$ when $t_1 > 0$, $\bar{t}$ increases monotonically, so that it is expressed uniquely in terms t_1.

Similarly, $A(\bar{t})$ is a unique function of t.

Now let,

$$\left.\begin{array}{ll} \delta^* = \text{displacement thickness,} = \displaystyle\int^{\infty} \left(1 - \frac{\rho u}{\rho_\infty U} \right) dy \\ \theta = \text{momentum thickness} = \displaystyle\int_0^{\infty} \left(1 - \frac{u}{U} \right) \frac{\rho u}{\rho_\infty U}\ dy \end{array}\right\} \qquad (23)$$

So that, $$\delta^* = \frac{c\sqrt{\bar{\nu}_0}\ \sqrt{3\bar{t}}}{N}\ \eta_0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (24)$$

where, $$\eta_0 = \underset{\eta \to \infty}{Lt}\ (\bar{\eta} - \bar{t}),\ \bar{\eta} = y\ \frac{N}{c\sqrt{\bar{\nu}_0}\ \sqrt{3\bar{t}}}$$

$$\theta = \frac{c\sqrt{\bar{\nu}_0}\ \sqrt{3\bar{t}}}{N} \int_0^{\infty} F_\eta\ (1 - F_\eta)\ d_\eta \qquad \ldots \qquad \ldots \qquad \ldots \qquad (25)$$

$$\text{Now, } A\ (\bar{t}) = \frac{1}{\eta_0^2}\ \frac{\delta^{*2}\ \frac{dN}{dt_1}}{\bar{\nu}_0 N} \text{ and } X'\ B\ (\bar{t}) = \frac{\delta^{*2}}{\bar{\nu}_0}\ \frac{Nx'}{c} = \frac{\delta^{*2}\ U_x}{\bar{\nu}_0} \quad \ldots \quad (26)$$

$$\therefore\ A\ (\bar{t}) + X'\ B\ (t) = \frac{\delta^{*2}\ (\bar{U}_{t_1} + \bar{U}\bar{U}_x)}{\nu_0\ U} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (27)$$

These give the relations between momentum and displacement thickness and the new variables.

Skin friction

The local shearing stress at the wall,

$$\tau_0(x) = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{\rho}{\rho_\infty} F_{\eta\eta} \ (x, 0, \bar{t}\ ) \frac{U(x, t)\ N\ (t_1)}{\sqrt{\bar{\nu}_0}\ \sqrt{3\bar{t}}} \quad .. \quad (28)$$

$$\text{Now,} \quad \frac{\mu \left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho N_0^2} = \sqrt{\nu}\ U\ (x, t) \frac{N(t_1)\ )}{N_0^2} \cdot \frac{1}{\sqrt{3\bar{t}}} F_{\eta\eta}\ (x, 0, \bar{t}\ ) \quad .. \quad (29)$$

From (22)

$$N\ (t_1) = N_0\, e^{\frac{1}{3}\int_{\bar{t}_0}^{\bar{t}} \frac{(A\bar{t})}{\bar{t}}\, d\bar{t}} = N_0 f\ (\bar{t})\ \text{(say)} \quad .. \quad (30)$$

$$\text{From } (A) \quad \frac{d\bar{t}}{dt_1} = \frac{N^2\ (t_1)}{C^2} = \frac{N_0^2}{C^2} f^2\ (\ \bar{t}\ ) \quad .. \quad .. \quad .. \quad (31)$$

$$\therefore \quad t_1 - t_{10} = \frac{C^2}{N_0^2} \int_{\bar{t}_0}^{\bar{t}} \frac{d\bar{t}}{f^2(\ \bar{t}\ )} \quad .. \quad .. \quad .. \quad (32)$$

$$t - t_0 = \frac{C}{N_0^2} \int_{\bar{t}_0}^{\bar{t}} \frac{d\bar{t}}{f^2(\ \bar{t}\ )}, \quad \bar{Y} = \frac{\sqrt{\bar{\nu}}\ \sqrt{3\bar{t}}}{N_0 f\ (\ \bar{t}\ )}\ \eta \quad .. \quad .. \quad (33)$$

Now, if $A(\ \bar{t}\ )$ be given, we have

$$u = N_0\, X\ (x)\, f\ (\ \bar{t}\ )\ F_\eta\ (n, \eta, \bar{t}\ ) \quad .. \quad .. \quad .. \quad (34)$$

$$\text{and} \quad t - t_0 \quad \frac{C}{N_0^2} \int_{\bar{t}_0}^{\bar{t}} \frac{d\bar{t}}{f^2(\ \bar{t}\ )}, \quad \bar{Y} = \frac{\sqrt{\bar{\nu}}\ \sqrt{3\bar{t}}}{N_0\, f(\bar{t}\ )}\, \eta \quad .. \quad .. \quad (35)$$

For practical problems, $N\ (t_1)$ will be given, whence we obtain $E = \bar{t}\ (t)$ by inversion, we get $t = t_1/c = t(\ \bar{t}\ )$ hence $A = A(\ \bar{t}\ )$

Let us first find the forms of $N(\ \bar{t}\ )$, when $A(\ \bar{t}\ )$ is given by

$$\left.\begin{array}{l} \text{(i)}\ A\ (\ \bar{t}\ ) = A_0 + A_1 \bar{t} + A_2 \bar{t}^{\,2} + \quad .. \quad .. \\ \text{(ii)}\ A(\ \bar{t}\ ) = A_0 + A_{1/3}\ \bar{t}^{\,1/3} + A_{2/3}\ \bar{t}^{\,2/3} + \quad .. \quad .. \end{array}\right\} \quad .. \quad (36)$$

From (30), we obtain,

$$\left.\begin{array}{l} \text{(i)}\ \ N\ (\ \bar{t}\ ) = N_0 \left(\dfrac{\bar{t}}{\bar{t}_1}\right)^{\frac{A_0}{3}} \left\{ l_0 + l_1 \bar{t} + l_2\ \bar{t}^2 + \quad .. \right\} \\ \text{(ii)}\ \ N\ (\ \bar{t}\ ) = N_0 \left(\dfrac{\bar{t}}{\bar{t}_0}\right)^{\frac{A_0}{3}} \left\{ l_0 + l_{1/3}\ \bar{t}^{\,1/3} + \quad .. \right\} \end{array}\right\} \quad (37)$$

Therefore from (32) & (37), we obtain,

$$\left.\begin{array}{l}\text{(i) } t_1(\bar{t}) = \bar{t}^{\,1-\frac{2}{3}A_0}\,[\,n_0 + n_1\bar{t} + \quad .. \quad ..\,] \\ \text{(ii) } t_1(\bar{t}) = \bar{t}^{\,1-\frac{2}{3}A_0}\,[\,n_0 + n_{1/3}\,\bar{t}^{\,1/3} + \quad ..\,]\end{array}\right\} \quad .. \quad (38)$$

From (38) inverting the series, and substituting for $\bar{t}$ in (37) we find,

$$\left.\begin{array}{l}\text{(i) } N(t_1) = t_1^{p}\,[\,r_0 + r_1\,t_1^{(2p+1)} + r_2\,t_1^{2(2p+1)} + \quad ..\,] \\ \text{(ii) } N(t_1) = t_1^{p}\,[\,r_0 + r_{1/3}\,t_1^{\frac{(2p+1)}{3}} + \quad ..\,]\end{array}\right\} \quad (39)$$

where, $\quad p = \dfrac{A_0}{3 - 2A_0}, \quad S_0 \quad A_0 = \dfrac{3p}{2p+1}, \quad p \neq 1/2\,.$

Particular case

When $A(\bar{t}) = A_0$ — In this case the form of velocity allows similarity solutions. The partial differential equations may be reduced to total ordinary differential equations:

From (30), we obtain,

$$N(t_1) = N_0\left(\frac{\bar{t}}{\bar{t}_0}\right)^{\frac{A_0}{3}} \quad .. \quad .. \quad (40)$$

From (32)

$$t_1 - t_{10} = \frac{C^2}{N_0^2}\int_{\bar{t}_0}^{\bar{t}}\left(\frac{\bar{t}_0}{\bar{t}}\right)^{\frac{2A_0}{3}} d\bar{t}$$

$$= \left\{\begin{array}{l}\dfrac{3\bar{\nu}\,\bar{t}_0}{N_0^2\,(3 - 2A_0)}\left[\left(\dfrac{\bar{t}_0}{\bar{t}}\right)^{\frac{2}{3}A_0 - 1} - 1\right] \text{ for } A_0 \neq 3/2 \\ \dfrac{\bar{\nu}\bar{t}_0}{N_0^2}\, ln\,\dfrac{\bar{t}}{\bar{t}_0} \text{ for } A_0 = 3/2.\end{array}\right\} \quad .. \quad (41)$$

Suitably choosing t_{10}, we may write these as,

$$t_1 = \left\{\begin{array}{l} t_{10}\left(\dfrac{\bar{t}_0}{\bar{t}}\right)^{\frac{2}{3}A_0 - 1} \text{ for } A_0 \neq 3/2 \\ t_{10}\left[\,1 + ln\left(\dfrac{\bar{t}}{\bar{t}_0}\right)\right] \text{ for } A_0 = 3/2\end{array}\right\} \quad .. \quad (42)$$

So, inverting,

$$\frac{\bar{t}}{\bar{t}_0} = \begin{cases} \left(\dfrac{t_1}{t_{10}}\right)^{\frac{3p}{3-2A_0}} & \text{for } A_0 \neq 3/2 \\ \exp.\left(\dfrac{t_1 - t_{10}}{t_{10}}\right) & \text{for } A_0 = 3/2 \end{cases} \quad .. \quad .. \quad (43)$$

From, (39), we get,

$$\mathcal{N}(t_1) = \begin{cases} \mathcal{N}_0 \left(\dfrac{t_1}{t_{10}}\right)^p = \mathcal{N}_0 \left(\dfrac{t}{t_0}\right)^p & \text{for } p \neq \infty \\ \mathcal{N}_0 \exp.\left(\dfrac{1}{3}\ \dfrac{t_1 - t_{10}}{t_{10}}\right) = \mathcal{N}_0 \exp.\left(\dfrac{1}{3}\ \dfrac{t - t_0}{t_0}\right) & \text{for } p = \infty \end{cases} \quad (44)$$

In the general case,

$$\left.\begin{aligned} &\text{(i)} \quad B(\bar{t}) = \bar{t}^k (B_0 + B_1 \bar{t} + B_2 \bar{t}^2 + \quad .. \quad .. \ ) \\ &\text{(ii)} \quad B(\bar{t}) = \bar{t}^k (B_0 + B_{1/3} \bar{t}^{1/3} + B_{2/3} \bar{t}^{2/3} + \quad .. \ ) \end{aligned}\right\} \quad .. \quad (45)$$

But in the case (44),

$$\left.\begin{aligned} B(\bar{t}) = B_0 \bar{t}^k \quad .. \\ \text{where } k = (p+1)/(2p+1) \end{aligned}\right\} \quad .. \quad (46)$$

When $\mathcal{N}(t_1)$ is given. Let, $\mathcal{N}(t_1)$ be analytical in the interval $0 \leqslant t < \infty$, and represented in terms of the convergent power series,

$$\mathcal{N}(t_1) = \sum_{k=0}^{\infty} \mathcal{N}_k t_1^k \quad .. \quad .. \quad .. \quad (47)$$

There may be two particular cases as follows:

$$\left.\begin{aligned} &\text{(i)} \quad \mathcal{N}(t_1) = \mathcal{N}_0 + \mathcal{N}_1 t_1 + D_2 t_1^2 + \quad .. \qquad \mathcal{N}_0 \neq 0 \\ &\text{(ii)} \quad \mathcal{N}(t_1) = \mathcal{N}_1 t_1 + \mathcal{N}_2 t_1^2 + \quad .. \qquad \mathcal{N}_1 \neq 0 \end{aligned}\right\} \quad .. \quad (48)$$

Case I: When $p = 0$—This represents that initially the body was moving with $\mathcal{N}_0$ when $t = 0$ and subsequently the change in $\mathcal{N}_0$ is given by (48i).

From (A') and (48i), we obtain

$$\bar{t} = \frac{1}{c^2}\left(\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{1}{k+i+1} \mathcal{N}_k \mathcal{N}_i t_1^{k+i+1}\right) \quad .. \quad .. \quad (49)$$

$$\bar{t} = q_1 t_1 + q_2 t_1^2 + \quad .. \quad .. \quad .. \quad (50)$$

and by inversion theorem

$$t_1 = p_1 \bar{t} + p_2 \bar{t}^2 + \quad .. \quad .. \quad .. \quad (51)$$

From (B) and (50)

$$\left.\begin{aligned} A(\bar{t}) &= c_1 t + c_2 t^2 + \\ B(\bar{t}) &= d_1 t + d_2 t^2 + \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

and so,

$$\left.\begin{aligned} &\text{(i)} \quad A(\bar{t}) = A_1 \bar{t} + A_2 \bar{t}^2 + \quad \ldots \quad = \sum_{i=0}^{\infty} A_i \bar{t}^i \\ &\text{(ii)} \quad B(\bar{t}) = B_1 \bar{t} + B_2 \bar{t}^2 + \quad \ldots \quad = \sum_{i=0}^{\infty} B_i \bar{t}^i \end{aligned}\right\} \quad \ldots \quad (53)$$

In (B), we substitute from (48i) and (A) and comparing with (53i) and equating different powers of t_1, we obtain,

$$\left.\begin{aligned} &A_0 = 0,\ A_1 = 3a_1,\ A_2 = -9a_1^2 + 6a_2 \\ &A_3 = 27a_1^3 - 33a_1 a_2 + 9a_3 \\ &A_4 = -81a_1^4 + 142a_1^2 a_2 - 54 a_1 a_3 - 22 a_2^2 + 12 a_4 \\ &A_5 = 243 a_1^5 + \frac{2415}{10} a_1^2 a_3 + \frac{988}{5} a_1 a_2^2 \end{aligned}\right\} \quad \ldots \quad (54)$$

... ...

$$\left.\begin{aligned} &B_0 = 0,\quad B_1 = 3b_1,\ B_2 = -3b_2 \\ &B_3 = -3b_3 + 6b_1 a_1^2,\ B_4 = -3b_4 + 14b_2 a_2 - 14b_1 a_1^3 \\ &B_5 = -3b_5 + \frac{33}{2} b_4 a_1 - \frac{105}{2} b_3 a_1^2 + 7 b_1 a_2^2 + 35 b_1 a_1^4 \end{aligned}\right\} \quad (55)$$

... ...

where

$$\left.\begin{aligned} a_i &= N_i C^{2i} \\ b_i &= \frac{N_{i-1} C^{2i}}{N_0^{2i}} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (56)$$

Case II : when $p = 1$—This follows if we take $p = 1$, in 2nd relation of (48). From (A) and (48ii), we obtain

$$\bar{t} = \frac{1}{C^2} \left(\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k+i+1} N_k N_i t_1^{k+i+1} \right) \quad \ldots \quad \ldots \quad (57)$$

and so $\bar{t} = q_1 t^3 + q_2 t^4 +$ .. .. .. (58)

Inverting the series, we obtain,

$$t = p_{1/3}\, \bar{t}^{1/3} + p_{2/3}\, \bar{t}^{2/3} + \quad \cdots \qquad (59)$$

The functions $A(\bar{t})$ and $B(\bar{t})$ will have the following form,

$$\left.\begin{aligned} A(\bar{t}) &= 1 + C_1 t_1 + C_2 t_1^2 + \\ B(\bar{t}) &= d_2 t_1^2 + d_3 t_1^3 + \end{aligned}\right\} \qquad (60)$$

Applying (59), we get,

$$\left.\begin{aligned} A(\bar{t}) &= 1 + A_{1/3}\, \bar{t}^{1/3} + \quad \cdots = \sum_{k=0}^{\infty} A_{k/3}\, \bar{t}^{k/3} \\ B(\bar{t}) &= B_{2/3}\, \bar{t}^{2/3} + B_1 \bar{t} \quad \cdots = \sum_{k=2}^{\infty} B_{k/3}\, \bar{t}^{k/3} \end{aligned}\right\} \qquad (61)$$

Proceeding as in the case-I, we get,

$$\left.\begin{aligned} A_0 &= 1,\ A_{1/3}\ \frac{1}{2} a_{1/3},\quad A_{2/3} = -\frac{23}{20} a_{1/3}^2 + \frac{6}{5} a_{2/3} \\ A_1 &= \frac{203}{120} a_{1/3}^3 - \frac{296}{30} a_{1/3} a_{2/3} - \frac{4}{3} a_1 \\ A_{4/3} &= -\frac{919}{600} a_{1/3}^4 + \frac{5226}{150} a_{1/3}^2 a_{2/3} + \frac{9}{7} a_{1/3} a_1 - \frac{1088}{175} a_{2/3}^2 - \frac{22}{21} a_{4/3} \\ A_{5/3} &= -\frac{1839}{7200} a_{1/3}^5 - \frac{2931}{30} a_{1/3}^3 a_{2/3} + \frac{2004}{735} a_{1/3}^2 a_1 + \frac{9576}{175} a_{1/3} a_{2/3}^2 \\ &\quad + \frac{539}{583} a_{1/3} a_{4/3} - \frac{39}{20} a_{2/3} a_1 - \frac{3}{4} a_{5/3} \end{aligned}\right\} (62)$$

$$\left.\begin{aligned} B_0 &= 0,\ B_{1/3} = 0,\ B_{2/3} = b_{2/3},\ B_{4/3} = \frac{1}{2} b_{2/3} a_{1/3}^2 - \frac{3}{5} b_{4/3} \\ B_{5/3} &= -\frac{7}{30} b_{2/3} a_{1/3}^3 + \frac{37}{30} b_1 a_{2/3} - \frac{2}{3} b_{5/3} \end{aligned}\right\} \qquad (63)$$

where,
$$\left.\begin{aligned} a_{i/3} &= N_{i+1} (3C^2)^{i/3} / (N_i^{2i/3} + 1) \\ b_{i/3} &- N_{i-1} (3C^2)^{i/3} / N_i^{2i/3} \end{aligned}\right\} \qquad (64)$$

Solution of the Partial Differential Equations

Case I: *When* $N(t_1) = \sum_{k=0}^{\infty} N_k t_1^k$ (*then* $A_0 = 0$, $p = 0$) We attempt a set of solutions of the form

$$\left.\begin{aligned} F_0(\eta, \bar{t}) &= \sum_{k=0}^{\infty} f_k(\eta)\, \bar{t}^k \\ F_1(\eta, \bar{t}) &= \sum_{k=1}^{\infty} g_k(\eta)\, \bar{t}^k \\ F_2(\eta, \bar{t}) &= \sum_{k=2}^{\infty} h_k(\eta)\, \bar{t}^k \\ F_{12}(\eta, \bar{t}) &= \sum_{k=1}^{\infty} I_k(\eta)\, \bar{t}^k \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (65)$$

Substituting in the differential equations (17) to (20), and equating the coefficients of various powers of $\bar{t}$ we obtain, (remembering in this case

$$\frac{p_1(k, t)}{p_\infty} = a_0(t) + a_1(t)\, X + a_2(t)\, X' +$$

where, $a_0(\bar{t}) = \sum_0^\infty b_k \bar{t}$, $a_1(\bar{t}) = \sum_0^\infty C_k \bar{t}$,

$a_2(\bar{t}) = \sum_0^\infty b_k \bar{t}$)

$$b_0 f_0''' + A_0 [1 - f_0' - \eta f_0''] + \frac{3}{2} \eta f_0'' = 0 \quad \ldots \quad \ldots \quad \ldots \quad (66)$$

$$\sum_{i=0}^{k} b_i f'''_{k-i} + A_k \{1 - f_0' - \eta f_0''\} - \sum_{i=0}^{k-1} A_i (f'_{k-i} + \eta f''_{k-i})$$
$$+ \frac{3}{2} \eta f_k'' - 3 k f_k' = 0, \; k = 1, 2, \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

... ...

$$b_0 g_1''' + C_1 f_0''' + A_0 \{-g_1' - \eta g_1''\} + \frac{3}{2} \eta g_1'' - 3 g_1 = 0 \quad \ldots \quad (68)$$

$$\sum_{i=0}^{k-1} b_i g'''_{k-i} + \sum_{i=1}^{k} C_i f'''_{k-i} - \sum_{i=0}^{k-1} A_i [g'_{k-1} + \eta g''_{k-1}] + \frac{3}{2} \eta g''_k - 3 k g_k'$$
$$= 0, \; k = 2, 3 \quad \ldots \quad \ldots \quad \ldots \quad (69)$$

... ...

$$\sum_{i=0}^{k-2} b_i h'''_{l-i} + \sum_{i=1}^{k} C_i g'''_{k-i+1} - \sum_{i=0}^{k-2} A_i \{h'_{k-i} + \eta h''_i\} + \frac{3}{2}\eta h''_k - 3^k h_k'$$
$$= 0, k = 2, 3, \quad .. \quad .. \quad .. \quad (70)$$

... ...

$$\sum_{i=0}^{k-1} b_i I'''_{k-i} + \sum_{i=1}^{k} d_i f'''_{k-i} + \sum_{i=0}^{k} A_i \{- I_{k-i} - \eta I''_{k-i}\} + \frac{3}{2}\eta I_k''$$
$$- 3 k I_k' = 0 \quad .. \quad .. \quad .. \quad (71)$$

... ...

The boundary conditions are as follows

(i) $f_0'(0) = 0 = f_0(0); f_0'(\infty) = 1$, $f_k(0) = f_k'(0) = 0, f_k'(\infty) = 0$ $\Big\}$ $k = 1, 2,$ .. .. (72)

(ii) $g_k(0) = g_k'(0) = 0$, $g_k'(\infty) = 0$ $\Big\}$ $k = 1, 2$ .. ..

(iii) $h_k(0) = h_k'(0) = 0, h_k'(\infty) = 0, k = 2, 3$ .. .. ..

(iv) $I_k(0) = I_k'(0) = 0$, $I_k'(\infty) = 0$ $\Big\}$ $k = 1, 2,$ .. .. ..

Let us change the independent variable in the form viz:

$$\eta_1 = l_n, \quad \text{where} \quad l = \sqrt{\frac{3}{4b_0}} \quad .. \quad .. \quad .. \quad (73)$$

The equations reduce to the following set :—

$$f_0''' + 2\eta_1 f_0'' = 0 \quad .. \quad .. \quad .. \quad (74)$$

$$f_1''' + 2\eta_1 f_1'' - 4 f_1' = - b_1 f_0''' - \frac{4}{3} A_1 \{1 - l(f_0' + \eta_1 f_0'')\}. \quad (75)$$

In general,

$$f_k''' + 2\eta_1 f_k'' - 4k_k f_k' = -\frac{1}{b_0}\sum_{i=1}^{k} b_i f_{k-i}''' - \frac{4}{3} A_k \{1 - l(f_0' + \eta_1 f_0'')\}$$
$$+ \frac{\sqrt{3}}{2 b_0 \sqrt{b_0}} \sum_{i=1}^{k-1} A_i (f'_{k-i} + \eta_1 f''_{k-i}) = P^1_k(\eta_1) \text{ (say)} \quad .. \quad (76)$$

Similarly,

$$g_k''' + 2\eta_1 g_k'' - 4k g_k' = -\frac{1}{b_0}\sum_{i=1}^{k-1} b_i g'''_{k-i} - \frac{1}{b_0}\sum_{i=1}^{k} C_i f'''_{k-i}$$

$$+ \frac{\sqrt{3}}{2\, b_0 \sqrt{b_0}} \sum_{i=1}^{k-1} A_i \left(g'_{k-i} + \eta_1 g''_{k-i} \right) = P^2_k (\eta_1) \text{ (say)} \quad .. \quad (77)$$

$$h'''_k + 2\eta_1 h''_k - 4k\, h'_k = -\frac{1}{b_0} \sum_{i=1}^{k-2} b_i\, h'''_{k-i} - \frac{1}{b_0} \sum_{i=1}^{k} C_i\, g'''_{k-i+1}$$

$$+ \frac{\sqrt{3}}{2\, b_0 \sqrt{b_0}} \sum_{i=1}^{k-2} A_i \{ h'_{k-i} + \eta_1 h''_i \} = P_k{}^3 (\eta_1) \text{ (say)} \quad .. \quad (78)$$

... ...

$$I'''_k + 2\eta_1 I''_k - 4_k I'_k = -\frac{1}{b_0} \sum_{i=1}^{k-1} b_i\, I'''_{k-i} - \frac{1}{b_0} \sum_{i=1}^{k} d_i\, f'''_{k-i}$$

$$+ \frac{\sqrt{3}}{2\, b_0 \sqrt{b_0}} \sum_{i=1}^{k-2} A_i \{ h'_{k-i} + \eta_1 h''_i \} = P^4_k (\eta_1) \text{ (say)} \quad .. \quad (79)$$

... ...

Now solutions for

$$y'' + 2x y' - 2n y = 0 \quad .. \quad .. \quad .. \quad (80)$$

is

$$y = A\, i^n\, erfc\, x + B\, i^n erfc\, (-x) \quad .. \quad .. \quad .. \quad (81)$$

(see Whittaker and Watson 1946).

where $i^n\, erfc\, x = \int_x^\infty i^{n-1}\, erfc\ t\, dt \ [\, n = 0, 1, 2, ..\ ]$

$$erfx = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$erfc\ x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - erf\, x$$

$$i^{-1}\, erfc\ x = \frac{2}{\sqrt{\pi}}\, e^{-x^2}$$, and $i^0\, erfc\ x = erfc\ x$

Now, $erf\,(-x) = 1 + erf\,(x)$; and as $\eta \longrightarrow \infty$

$f'(\infty) = 1$, we take $B = 0$

In terms of single integral,

$$i^n \; erfc \; x = \frac{2}{\sqrt{\pi}} \int\limits_x^{\infty} \frac{(t-x)^n}{n!} \; e^{-t^2} \; dt = p_n(x)$$

Putting here $n = 2k$, we may get the solution in the above form.

Now, solution for f_0, subjected to the boundary conditions (72) is as follows :—

$$f'_0(\eta_1) = 1 - p_0(\eta_1) \qquad \ldots \quad \ldots \quad \ldots \quad (82)$$

$$f_0(\eta_1) = \eta_1 + p_1(\eta_1) - \tfrac{1}{2} \cdot \frac{1}{\Gamma(3/2)}$$

$\left(\because \; i^n \; erfc \; 0 = \dfrac{1}{2^n \, \Gamma\left(\frac{n}{2}+1\right)} \right.$, $n = -1, 0, 1, \ldots$, Abramowitz Stegun 1964)

$$f_0''(\eta_1) = p_{-1}(\eta_1)$$

In general

$$f_k'(\eta_1) = A^k \, p_n(\eta_1) + \int\limits_{\eta_1}^{\infty} \frac{P_k^1(\xi)\,[p_n(\eta_1)\,p_{1n}(\xi) - p_n(\xi)\,p_{1n}(\xi_1)]\,d\xi}{\begin{vmatrix} p_n(\xi) & p_{1n}(\xi) \\ p'_n(\xi) & p'_{1n}(\xi) \end{vmatrix}}$$

$$\text{where } A^k = -\frac{2}{\sqrt{\pi}} \int\limits_{\eta_1}^{\infty} \frac{P_k^1(\xi)\,[\,p_n(\eta_1)\,p_{1n}(\xi) - p_n(\xi)\,p_{1n}(\eta_1)]\,d\xi}{\begin{vmatrix} p_n(\xi) & p_{1n}(\xi) \\ p'_n(\xi) & p'_{1n}(\xi) \end{vmatrix}}$$

[in view of the boundary conditions (72)].

where $\quad p_{1n} = i^n \; erfc\,(-\eta_1)$

Similarly,

$$g'_k(\eta_1) = G_k \, p_n(\eta_1) + \int\limits_{\eta_1}^{\infty} \frac{P_k^2(\xi)\,[p_n(\eta_1)\;p_{1n}(\xi) - p_n(\xi)\;p_{1n}(\eta_1)]\,d\xi}{\begin{vmatrix} p_n(\xi) & p_{1n}(\xi) \\ p^1_n(\xi) & p'_{1n}(\xi) \end{vmatrix}} \quad \ldots \quad (84)$$

$$G_k = -\frac{2}{\sqrt{\pi}} \int\limits_0^{\infty} \frac{P_k^2(\xi)\,[p_n(\eta_1)\,p_{1n}(\xi) - p_n(\xi)\,p_{1n}(\eta_1)]\,d\xi}{\begin{vmatrix} p_n(\xi) & p_{1n}(\xi) \\ p'_n(\xi) & p'_{1n}(\xi) \end{vmatrix}}$$

and

$$h'_k(\eta_1) = H_k\, p_n(\eta_1) + \int_{\eta_1}^{\infty} \frac{P_k^3(\xi)\,[p_n(\eta_1)\,p_{1n}(\xi) - p_n(\xi)\,p_{1n}(\eta_1)]\,d\xi}{\begin{vmatrix} p_n(\xi) & p_{1n}(\xi) \\ p'_n(\xi) & p'_{1n}(\xi) \end{vmatrix}} \quad \ldots \quad (85)$$

$$H_k = -\frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{P_k^3(\xi)\,[p_n(\eta_1)\,p_{1n}(\xi) - p_n(\xi)\,p_{1n}(\eta_1)]\,d\xi}{\begin{vmatrix} p_n(\xi) & p_{1n}(\xi) \\ p'_n(\xi) & p'_{1n}(\xi) \end{vmatrix}}$$

... ...

$$I'_k(\eta_1) = E_k\, p_n(\eta_1) + \int_{\eta_1}^{\infty} P_k^4(\xi)\,[p_n(\eta_1)\,p_{1n}(\xi) - p_n(\xi)\,p_{1n}(\eta_1)]\Big/\begin{vmatrix} p_n & p_{1n} \\ p'_n & p'_{1n} \end{vmatrix}\,d\xi \quad \ldots \quad (86a)$$

and so on.

Case II : <u>*When* $A_0 = 1 \quad (p = 1)$ — Then</u>

$$N(\bar{t}) = \sum_{k=0}^{\infty} N_{k+1}\; E_1^{k+1} \qquad (N_1 \neq 0)$$

$$A(\bar{t}) = \sum_{k=0}^{\infty} A_{k/3}\; \bar{t}^{\,k/3}\;, \quad B(\bar{t}) = \sum_{k=2}^{\infty} B_{k/3}\; \bar{t}^{\,k/3}$$

Now forms of $A(\bar{t})$, $B(\bar{t})$, suggest that a set of solutions in the form may be possible :—

$$F_0(\eta, \bar{t}) = \sum_{k=0}^{\infty} f_{k/3}(\eta)\; \bar{t}^{\,k/3} \quad \ldots \quad \ldots \quad \ldots \quad (86)$$

$$F_1(\eta, \bar{t}) = \sum_{k=1}^{\infty} g_{k/3}(\eta)\; \bar{t}^{\,k/3} \quad \ldots \quad \ldots \quad \ldots \quad (87)$$

$$F_2(\eta, \bar{t}) = \sum_{k=2}^{\infty} h_{k/3}(\eta)\; \bar{t}^{\,k/3} \quad \ldots \quad \ldots \quad \ldots \quad (88)$$

$$F_{12}(\eta, \bar{t}) = \sum_{k=2}^{\infty} I_{k/3}(\eta)\; \bar{t}^{\,k/3} \quad \ldots \quad \ldots \quad \ldots \quad (89)$$

Also, in this case, possible forms of $a_0(t)$, $a_1(t)$, $a_i(t)$ are as follows:—

$$a_0(t) = \sum_{k=0}^{\infty} b_{k/3}\; \bar{t}^{\,k/3}\;, \quad a_1(t) = \sum_{k=1}^{\infty} c_{k/3}\; \bar{t}^{\,k/3}\;, \quad a_i(t) = \sum_{k=2}^{\infty} d_{k/3}\; \bar{t}^{\,k/3}$$

Substituting these in the equations (17) to (20) and equating to zero, different powers of $\bar{t}$, we obtain,

$$b_0 f_0''' + A_0 [1 - f_0' - \eta f_0''] + \frac{3}{2} \eta f_0'' = 0$$

$$\sum_{i=0}^{k} b_{i/3} f'''_{\frac{k-i}{3}} + A_k [1 - f_0' - \eta f''_0] + \frac{3}{2} \eta f''_{k/3} - k f'_{k/3}$$
$$- \sum_{i=0}^{k-1} A_{i/3} (f'_{\frac{k-i}{3}} + \eta f''_{\frac{k-i}{3}}) = 0 \quad .. \quad .. \quad .. \quad (90)$$

... ...

$$\sum_{i=0}^{k-1} b_{i/3} g'''_{\frac{k-i}{3}} + \sum_{i=1}^{k} C_{i/3} f'''_{\frac{k-i}{3}} - \sum_{i=0}^{k-1} A_{i/3} \{g'_{\frac{k-i}{3}} + \eta g''_{\frac{k-i}{3}}\}$$
$$+ \frac{3}{2} \eta g''_{k/3} - k g'_k = 0, k = 1, 2 \quad .. \quad .. \quad .. \quad (91)$$

... ...

$$\sum_{i=0}^{k-2} b_{i/3} h'''_{\frac{k-i}{3}} + \sum_{i=1}^{k-i} C_i g'''_{\frac{k-i}{3}} - \sum_{i=0}^{k-2} A_{i/3} (h'_{\frac{k-i}{3}} + \eta h''_{\frac{k-i}{3}}) + \frac{3}{2} \eta h''_{k/3}$$
$$- k h'_{k/3} = 0, k = 2, 3, \quad .. \quad .. \quad .. \quad .. \quad (92)$$

... ...

$$\sum_{i=0}^{k-2} b_{i/3} I'''_{\frac{k-i}{3}} + \sum_{i=2}^{k} d_{i/3} f'''_{\frac{k-i}{3}} - \sum_{i=0}^{k-2} A_{i/3} (I'_{\frac{k-i}{3}} + \eta I''_{\frac{k-i}{3}}) + \frac{3}{2} \eta I_{k/3}$$
$$- k I'_{k/3} + B_{k/3} \{1 - f_0'^2 - f_0 f_0''\}$$
$$+ \sum_{k=2}^{k-1} B_{i/3} \left[- \sum_{j=0}^{k-1} f_{i/3} f'_{\frac{k-i-j}{3}} + \sum_{j=0}^{k-i} f_{j/3} f''_{\frac{k-i-j}{3}} \right] = 0, k = 2, 3$$
$$.. \quad .. \quad .. \quad (93)$$

The boundary conditions are :

$$\left.\begin{array}{l} \text{(i)} \ f_0'(0) = 0, f_{k/3}(0) = f'_{k/3}(0) = 0, f'_0(\infty) = 1, f'_{k/3}(\infty) = 0, k = 1, 2 \\ \text{(ii)} \ g_{k/3}(0) = g'_{k/3}(0) = 0, g'_{k/3}(\infty) = 0, \ k = 1, 2, 3 \\ \text{(iii)} \ h_{k/3}(0) = h'_{k/3}(0) = 0, \ h'_{k/3}(\infty) = 0, \ k = 2, 3, \\ \text{(iv)} \ I_{k/3}(0) = I'_{k/3}(0) = 0, \ I'_{k/3}(\infty) = 0, \ k = 2, 3 \end{array}\right\} \quad (94)$$

As before we change the independent variable as follows:

$\eta_2 = l_2\, \eta$, where, $l_2 = \sqrt{1/4b_0}$.

The equations becomes :- (here $A_0 = 1$)

$$f'''_{k/3} + 2\eta_2 f''_{k/3} - 4\,(k+1)\, f'_{k/3} = p_k^{1/3}\,(\eta_2) \quad \text{(say) } k = 0, 1, 2, \qquad (95)$$

$$g'''_{k/3} + 2\eta_2\, g''_{k/3} - 4\,(k+1)\, g'_{k/3} = P_k^{2/3}\,(\eta_2) \quad \text{(say) } k = 1, 2,$$

Putting $2\,(k+1) = n$, a solution can be given in the form:

$$f'_{k/3}\,(\eta_2) = B^{k/3}\, p_n^{1/3}\,(\eta_2) + \int_{\eta_2}^{\infty} \frac{p_k^{1/3}\,(\xi)\,[p_n\,(\eta_2)\, p_{1n}\,(\xi) - p_n\,(\xi)\, p_{1n}\,(\eta_2)]\, d\xi}{\begin{vmatrix} p_n\,(\xi) & p_{1n}\,(\xi) \\ p'_n\,(\xi) & p'_{1n}\,(\xi) \end{vmatrix}} \qquad (96)$$

where $$B^{k/3} = -\frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{P_k^{1/3}\,(\xi)\,[p_n\,(\eta_2)\, p_{1n}\,(\xi) - p_n\,(\xi)\, p_{1n}\,(\eta_2)]\, d\xi}{\begin{vmatrix} p_n\,(\xi) & p_{1n}\,(\xi) \\ p'_n\,(\xi) & p'_{1n}\,(\xi) \end{vmatrix}}$$

and $p_n\,(z) = i^n\, erfc,\,(z)$, $p_{1n}\,(z) = i^n\, erfc\,(-z)$

... ...

$$g'_{k/3}\,(\eta_2) = G^{k/3}\, p_n\,(\eta_2) + \int_{\eta_2}^{\infty} \frac{P_k^{2/3}\,(\xi)\,[p_n\,(\eta_2)\, p_{1n}\,(\xi) - p_n\,(\xi)\, p_{1n}\,(\eta_2)]\, d\xi}{\begin{vmatrix} p_n\,(\xi) & p_{1n}\,(\xi) \\ p'_n\,(\xi) & p'_{1n}\,(\xi) \end{vmatrix}}$$

$$G^{k/3} = -\frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{p_k^{1/3}\,(\xi)\,[p_n\,(\eta_2)\, p_{1n}\,(\xi) - p_{1n}\,(\eta_2)\, p_n\,(\xi)]\, d\xi}{\begin{vmatrix} p_n\,(\xi) & P_{1n}\,(\xi) \\ p'_n\,(\xi) & p'_{1n}\,(\xi) \end{vmatrix}} \qquad (97)$$

and so on.

A Simple Application

Watson-type (1955) *Similarity Solution*

Let us consider the case, when a flat plate is accelerating through a compressible fluid, with velocity $U\,(t) = No\, t^p$ so that in this case $X\,(x) = 1$. The corresponding problem in an incompressible flow was investigated by Watson (1955).

To apply our theory, let us assume

$$N\,(t_1) = No\, t^p \quad \text{for} \quad t \leqslant t^1$$
$$= 0 \quad \text{for} \quad t > t'$$

(since in the present example, we may think of a missile flight in the primary stages of raphid acceleration).

Watson assumed this type of velocity distribution in the primary stages of boundary layer development. Obviously one cannot assume this type of velocity distribution for a very long time.

Therefore

$$C < \infty \quad \frac{p_e}{p_\infty} = 1$$

The functions $A(\bar{t})$ and $B(\bar{t})$ take the forms,

$$A(\bar{t}) = A_0 \quad \text{and} \quad B(\bar{t}) = B_k\ \bar{t}^k$$

where

$$B_k = \frac{3\mathcal{N}_0}{2p+1}\left[\frac{C^2(2p+1)}{\mathcal{N}_0^{\,2}}\right]^{(p+1)/(2p+1)} = \frac{3}{2p+1}\ B^*$$

$$\& \ B^* = \left[\frac{C^2(2p+1)}{\mathcal{N}_0^{\,2}}\right]^{\frac{p+1}{2p+1}}$$

$$\text{and} \quad K = \frac{(p+1)}{2p+1}$$

$$\therefore F_0(\eta, \bar{t}) = f(\eta),\ F_1(\eta, \bar{t}) = g(\eta)\ \bar{t},^k\ F_2(\eta, \bar{t}) = h(\eta)\ \bar{t}^{2k} \quad .. \quad (98)$$

Let us introduce the variables

$$\eta = \sqrt{(2p+1)/3}\ \eta_1,\ \bar{t} = \frac{1}{B^*}\ t^{2p+1},\ f = f_0,\ g = B^*\ g_0,\ h = B^*\ h_0$$

We obtain

$$f_0''' + 2\eta_1 f_0'' - 4(1 - f_0') = 0$$

$$g_1''' + 2\eta_1 g_0'' - 4(2p+1)\,g_0' = -4(1 - f_0'^2 + f_0 f_0'')$$

$$h_0''' + 2\eta_1 h_0'' - 4(3p+2)\,h'_0 = -4(-2f_0' g_0' + f_0 g''o + f_0'' g_0)$$

... ...

with the boundary conditions :

$$f_0(0) = f_0'(0) = 0\ ; \qquad g_0(0) = g'_0(0) = 0$$

$$h_0(0) = h_0'(0) = 0\ ; \qquad f'(\infty) = 1,\ h_0'(\infty) = 0,\ g_0'(\infty) = 0$$

These equations with the boundary conditions are identical, with those deduced by Watson (1955).

Advantage of the Present Method

Nobody has so far investigated the motion of a two-dimensional body through a compressible (viscous) fluid when the flight velocity is of the general form $X(x) N(t)$. Some has investigated the corresponding flow for a flat plate when $X(x) = 1$. Our work gives a generalized method of solution for a compressible boundary layer flow past a two-dimensional body (or a two dimensional body accelerating through a compressible fluid at rest at infinity), when $N(t)$ is square integrable. It also gives a procedure of investigating boundary layer development for a finite time in a compressible flow, when $N(t)$ is not square-integrable. In that case we simply define it in the following manner:

$$\begin{aligned} N(t) &= f(t) \qquad && t \geqslant t' \\ &= 0 && t < t' \end{aligned}$$

where $(0, t')$ is the time interval of our discussion.

Separation

From (29), the position of separation at different moments of time may be calculated as follows :

$$F_{\eta\eta}(x, o, \bar{t}) = 0 \quad \text{which gives}$$

$$\sum_{k=0}^{\infty} f''_k(0)\, \bar{t}^k + X \sum_{k=1}^{\infty} g''_k(0)\, \bar{t}^k + X^2 \sum_{k=2}^{\infty} h''_k(0) \bar{t}^{\,k} + + X' \sum_{k=1}^{\infty} I''_k(0)\, \bar{t}^k + \quad .. \qquad .. \qquad .. = 0. \qquad .. \qquad (99)$$

In each particular case, we shall have a definite number of terms or dominant terms depending on the raphidity of convergence; stopping after these terms (99), will give the position of separation for different moments of tiцme.

From the first appearance of separation, the point of separation will move along the contour to attain a steady state.

Skin Friction

From (29),skin-friction is given as follows:

$$\frac{\mu \left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho N_0^2} = \sqrt{\nu}\, U(x, t)\ \frac{N(t_1)}{N_0^2 \sqrt{3\,\bar{t}}} \Bigg[\sum_{k=0}^{\infty} f_0''(0)\, \bar{t}^k + X(x) \sum_{k=1}^{\infty} g''_k(0)\, \bar{t}^k + X^2 \sum_{k=1}^{\infty} h''_k(0)\, \bar{t}^k + X'(x) \sum_{k=1}^{\infty} I''_k(0)\, \bar{t}^k + \quad .. \quad .. \Bigg]$$

Displacement thickness

$$\delta^* = \frac{C\sqrt{\nu_0}\sqrt{3\bar{t}}}{N}\left\{ at_{\eta\to\infty}(\eta - f_0) - \left[\sum_{k=1}^{\infty} f_k(\infty)\bar{t}^k + X(x)\sum_{k=1}^{\infty} g_k(\infty)\bar{t}^k + X'(x)\sum_{k=2}^{\infty} h_k(\infty)\bar{t}^k + X^2(x)\sum_{k=1}^{\infty} I_k(\infty)\bar{t}^k + \ldots \ldots \right]\right\}$$

Velocity

$$u = N(t_1)X(x)\left[\sum_{k=0}^{\infty} f_k'(\eta)\bar{t}^k + X(x)\sum_{k=1}^{\infty} g_k(\eta)\bar{t}^k + X'(x)\sum_{k=2}^{\infty} h_k(\eta)\bar{t}^k + X^2(x)\sum_{k=1}^{\infty} I_k(\eta)\bar{t}^k + \ldots \ldots \right]$$

and so on, for other physical variables.

Temperature Variable

Though solution for temperature variable for any Pr requires detailed discussions, which will be presented afterwards, in the present part, a procedure for its determination is indicated when

$$Pr = 1.$$

Following Howarth, we assume here $Pr = 1$, so that we can use the well-known solution of the energy equation, in the form:

$$C_p T + \frac{1}{2} u^2 = \text{constant.}$$

This solution is valid if $P_r = 1$ whatever be the pressure distribution in the main-stream and regardless of the dependence of viscosity on temperature.

The constant may be evaluated, from the conditions in the main-stream as follows:

$$\frac{T}{T\infty} = 1 + \frac{\gamma - 1}{2a_\infty^2}(U^2 - u^2)$$

where a_∞ is the velocity of the sound in the main-stream, γ is the ratio of the specific heats.

Appendix I

Convergence of the series for $F(x, \eta, \bar{t})$:

Now, $F(x, \eta, \bar{t}) = F_0(\eta, \bar{t}) + X F_1(\eta, \bar{t}) + X^2 F_2(\eta, \bar{t}) + X' F_{12}(\eta, \bar{t})$

$$= \sum_{k=0}^{\infty} [f_k(\eta)\, \bar{t}^k + \{X g_{k+1} + X' I_{k+1}\}\, \bar{t}^{k+1} + X^2 h_{k+2}\, \bar{t}^{k+2} + \quad .. \quad ..]$$

From the definition of $\bar{t}$, we find $\bar{t} < 1$. We assume $\mathcal{N}(t_1)$ to be differentiable, so that it is bounded.

Now, f_k, g_{k+1}, h_{k+2}, $\dot{I}_{k+2}$ .. .. .. are bounded.

We suppose

$$f_k \leqslant do,\; g_{k+1} \leqslant \frac{d_1}{2},\; h_{k+2} \leqslant d_2,\; I_{k+2} \leqslant \frac{d_{12}}{2} \quad .. \quad ..$$

$$\therefore\; F(x, \eta, \bar{t}) \leqslant \sum_{k=0}^{\infty} \bar{t}^k \left[do + (x + x') \frac{d_1}{2} \bar{t} + X^2 d_2 \bar{t}^2 + \ldots \right]$$

We have assumed that $X(x)$ possesses derivatives of all orders. Therefore, the derivatives form a set of bounded functions in some interval, say I, so that

$$\operatorname{Sup} \mid X^{(\eta)} \mid = K_\eta \;\therefore\; \text{ if } \operatorname{Sup}_x \int dx = b$$

we have

$$\operatorname{Sup}_{x \in I} \left| X^{(n-m)} \right| \leqslant \frac{k_n b^m}{m!}$$

We assume $X > X'$, $X^2 > XX'$, $X^3 > XX' X'' > XX''$ etc.

$$\therefore \quad F < \sum_{k=0}^{\infty} \bar{t}^k \left[do + \sum_{j=1}^{n} d_j (x\, \bar{t})^j \right]$$

$$\therefore \quad F < \sum_{k=0}^{\infty} \bar{t}^k \left[do + \sum_{j=1}^{n} d_j \frac{(D\, \bar{t})}{\{\eta!\}^j} \right]$$

where $D = K_n b^\eta$

The bracketed series is convergent as $\eta \to \infty$ for all D and $\bar{t}$. The whole series is convergent if $\bar{t} < 1$. By our choice $\bar{t}$ is always < 1. Therefore, we arrive at a satisfactory conclusion that our analysis is valid for all finite t.

A short note on the method of application of the present theory

Usually in the problems of the boundary layer theory, the form of the velocity in the main-stream is prescribed, so that $\mathcal{N}(t_1)$ is known beforehand. Hence we can easily calculate $A(\bar{t})$. A procedure when main-stream velocity is prescribed is formulated below :

$$\mathcal{N}(t_1) = \mathcal{N}_0 + \mathcal{N}_1 t_1 + \mathcal{N}_2 t_1^2 + \mathcal{N}_3 t_1^3 + \quad .. \quad .. \quad ..$$

(1) Calculation of the transition of coordinates from Υ. t to

$$\eta = \frac{\Upsilon \mathcal{N}(\bar{t})}{C\sqrt{\nu_0}\sqrt{3\bar{t}}} \quad \text{and} \quad \bar{t} = \frac{1}{C^2}\int_0^t \mathcal{N}^2(t_1)\,dt$$

Now from (54) and (55), it is easy to calculate A_k and B_k so $A(\bar{t})$ and $B(\bar{t})$ may be evaluated in a straightforward manner.

(2) We can readily obtain

$$\frac{\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho \mathcal{N}_0^2} = \sqrt{\nu_0}\, U(x,t)\, \frac{\mathcal{N}(t_1)}{\mathcal{N}_0^2\sqrt{3\bar{t}}} \left[\sum_{k=0}^{\infty} f''_k(0)\,\bar{t}^k + \right.$$

$$\left. + X(x)\sum_{k=1}^{\infty} g''_k(0)\,\bar{t}^k + X^2 \sum_{k=2}^{\infty} h''_k(0)\,\bar{t}^k + X'(x)\sum_{k=1}^{\infty} I''_k(0)\,\bar{t}^k + \quad .. \quad .. \right]$$

(3) Displacement thickness may be obtained from the expression

$$\partial^* = \frac{C\sqrt{\nu_0}\sqrt{3\bar{t}}}{\mathcal{N}(t_1)} \left\{ at_{\eta\to\infty}(\eta - f_0) - \left[\sum_{k=1}^{\infty} f_k(\infty)\,\bar{t}^k \right.\right.$$

$$\left.\left. + X(x)\sum_{k=1}^{\infty} g_k(\infty)\,\bar{t}^k + X'(x)\sum_{k=2}^{\infty} h_k(\infty)\,\bar{t}^k + X^2(x)\sum_{k=1}^{\infty} I_k\,\bar{t}^k + \quad .. \quad .. \right]\right\}$$

(4) Velocity profiles at different places along the contour, at different moments of time is calculated from the formulae

$$u(x,y,\bar{t}) = \mathcal{N}(t)\, X(x) \left[\sum_{k=0}^{\infty} f'_k(\eta)\,\bar{t}^k + X(x)\sum_{k=1}^{\infty} g'_k(\eta)\,\bar{t}^k \right.$$

$$\left. + X'(x)\sum_{k=2}^{\infty} h'_k(\eta)\,\bar{t}^k + X^2(x)\sum_{k=1}^{\infty} I'_k(\eta) + \quad .. \quad .. \right]$$

(5) To find the separation moment we use the expression

$$\sum_{k=0}^{\infty} f''_k(0)\, \bar{t}^k + X \sum_{k=1}^{\infty} g''_k(0)\, \bar{t}^k + X^2 \sum_{k=2}^{\infty} h''_k(0)\, \bar{t}^k + X' \sum_{k=1}^{\infty} I''_k(0)\, \bar{t}^k + \\ + \quad .. \quad .. \quad = 0$$

and so on for other characteristics.

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