ON FIXED POINT THEOREMS FOR PRODUCT OF METRIC SPACES

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In the present paper, we have established certain fixed point theorems for product metric spaces.

INTRODUCTION

The well known contraction mapping theorem is concerned with mapping of a complete metric space into itself. This principle has been generalized in various ways by many authors viz. Rakotch (1962), Kannan (1968), Hardy and Rogers' (1973), Edelstein (1961) and others. In another paper, Edelstein (1962) established a fixed point therorem on a compact metric space and showed the existence of a unique fixed point of a contractive map defined on it. Further, Cohen (1973) discussed results on a product of metric spaces. He proved the following :

Theorem A—A CIEVS (contractive in each variable separately) function on a compact metric space $(X \times Y, \sigma)$ into itself must have a fixed point where σ is defined as

$$\sigma(\ (x_1, y_1),\ (x_2, y_2)\ ) = d_1\ (x_1,\ x_2) + d_2\ (y_1, y_2)$$

and d_1 and d_2 are metrics on X and Y respectively. Other metrics which are generally used on the product spaces are u and δ definied as

$$u\ ((x_1, y_1),\ (x_2, y_2)) = \max\ \{d_1\ (x_1,\ x_2),\ d_2\ (y_1, y_2)\}$$

and $$\delta\ ((x_1, y_1),\ (x_2, y_2)) = \{[d_1\ (x_1,\ x_2)]^2 + [d_2\ (y_1, y_2)]^2\}^{1/2}$$

In this paper, we have taken f^n to be contractive and CIEVS functions respectively and have extended results due to Edelstein (1962) and Cohen (1973). Suitable examples are also provided.

Throughout this paper, we shall confine to the notations given in the introduction

RESULT WITH CONTRACTIVE ITERATIE.

In this section, we generalize a theorem due to Edelstein (1962).

Theorem 1—Let f be a mapping of a compact metric space (X, d) into itself and let for some positive integner n, f^n be contractive. Then f has a unique fixed point in X.

PROOF : If the contractive map f^n moves all of the points of X then by compactness there exists a point p such that $d(f^n(p). p)$ is minimal.
But $d(f^n(p), f^n(f^n(p))) < d(p, f^n(p))$,
which contradicts the minimality of $d(p, f^n(p))$.
Therefore, we must have $d(p, f^n(p)) = 0 \Longleftrightarrow f^n(p) = p$.
Now if $q \in X$ is sueh that $f^n(q) = q$ then $d(p, q) = d(f^n(p), f^n(q)) < d(p, q)$, implying thereby that p is a unique fixed point of f^n. That p is a fixed point of f is as follows :

$$\begin{aligned} p = f^n(p) \Rightarrow f(p) &= f(f^n(p)) \\ &= f^n(f(p)) \\ \Rightarrow p = f(p). \end{aligned}$$

Lastly, uniqueness of p as a fixed point of f may be easily verified.

Theorem 1 is an extension of the result of Edelstein 1962. This can be observed from the fact that even if f^n is contractive, f need not be contractive.

Let $f : [0, 1] \rightarrow [0, 1]$ be defined by

$$f(x) = \begin{cases} 0 & , x \in [0, \frac{1}{2}] \\ \frac{1}{2} & , x \in (\frac{1}{2}, 1) \end{cases}$$

Then $f^2(x) = 0$ for all $x \in [0, 1]$ and so f^2 is contractive while f is not contractive.

RESULTS WITH CIEVS INTERATES

In this section, we consider f^n to be a CIEVS function on a product metric space and extend results due to Cohen (1973).

Theorem 2—Let f^n be a CIVES function on a compact metric space $(X \times Y, d)$ into itself. Then f has unique fixed point, where d is σ.

PROOF : We have

$$
\begin{aligned}
d((x_1, y_1), (x_2, y_2)) &= d_1(x_1, x_2) + d_2(y_1, y_2) \\
&= d((x_1, y_1), (x_2, y_1)) + d((x_2, y_1), (x_2, y_2)) \\
&> d(f^n(x_1, y_1), f^n(x_2, y_1)) + d(f^n(x_2, y_1), f^n(x_2, y_2)) \\
&\geqslant d(f^n(x_1, y_1), f^n(x_2, y_2)).
\end{aligned}
$$

Therefore f^n is contractive and by Theorem 1 has a unique fixed point.

That the preceding theorem is indeed an extension of Theorem A, can be seen from the following example.

Consider a function

$$f : [0, 1] \times [0, 1] \longrightarrow [0, 1] \times [0, 1]$$

defined as follows :

$$
f(x, y) = \begin{cases}
(0, 0), & x \in [0, \tfrac{1}{2}], \; y \in [0, \tfrac{1}{2}] \\
(0, \tfrac{1}{2}), & x \in [0, \tfrac{1}{2}], \; y \in (\tfrac{1}{2}, 1) \\
(\tfrac{1}{2}, 0), & x \in (\tfrac{1}{2}, 1), \; y \in [0, \tfrac{1}{2}] \\
(\tfrac{1}{2}, \tfrac{1}{2}), & x \in (\tfrac{1}{2}, 1), \; y \in (\tfrac{1}{2}, 1)
\end{cases}
$$

Since $f^2(x, y) = (0, 0)$ for all $x, y \in [0, 1]$,therefore f^2 is contractive and hence is a CIEVS function. But f is not contractive in the first variable. It can be observed by taking

$$x_1 = \frac{1}{2} \text{ and } x_2 = \frac{3}{4}.$$

Consequently f is not CIEVS.

Cohen (1973, Theorems 3, 5) has shown the existence of a fixed point (not necessarily unique) of a CIEVS function f. By taking f^n to be CIEVS in place of f one may not guarantee a fixed point of f. In the following, we make certain observations in this direction as to when the existence of a unique fixed point of f can be assured.

Theorem 3—Let f^n be CIEVS on the compact space $(X \times Y, d)$ and let the space (X, d_1) (or (Y, d_2)) have the fixed point property, d being μ or δ. Then (Cohen, 1973, Theorem 3) f^n has a fixed point (x_0, y_0) in $X \times Y$. This as such does not assure the existence of a fixed point of f. However if (x_0, y_0) is unique as a fixed point of f^n then it can be readily seen that (x_0, y_0) is a fixed point of f and is unique.

Let S' denote the unit circle in the complex plane and $d = \mu$ or δ. We have the following :

Theorem 4—If f^n is a CIEVS map on $S' \times S'$ then (Cohen 1973, Theorem 5) f^n has a fixed point. Further if this fixed point is unique then it will also be a fixed point of f and unique.

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