A FIXED POINT THEOREM CONCERNING ITERATES OF A MAPPING

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In a recent paper, the authors discussed continuous self-mappings of a complete metric space satisfying a set of conditions and showed the existence of a unique fixed point for such mappings. In the present paper we generalize that result to mappings which need not be continuous but satisfy the same set of conditions at each point in a subset of the space.

§ 1. The well known contraction mapping theorem runs as follows :

Every contraction mapping of a complete metric space (X, d) into itself has a unique fixed point (Brown and Page 1970). Various generalisations of this result have been made by Belluce and Kirk (1969), Edelstein (1961), Sehgal (1969) and others. Sehgal (1969) investigated mappings of a complete metric space with a contractive iterate at each point of the space and showed that such mappings have a unique fixed point and the sequence of iterates of any point in the space converges to the fixed point. This result has been generalized by Guseman (1970) to mappings which are not necessarily continuous and which have a contractive iterate at each point in a subset of the space. Khazanchi and Dass (1976) established the existence of a unique fixed point on the lines of Sehgal (1969) for continuous mappings satisfying a set of conditions. The idea of considering this set of conditions was initiated from a paper by Zamfirescu (1972).

The object of this paper is to extend the result obtained by authors (Khazanchi and Dass 1976) to mappings which are not necessarily continuous and which satisfy a set of conditions in a subset of the space. The line of approach is analogous to that of Guseman (1970).

§ 2. We prove the following :

Theorem—Let f be a mapping of a complete metric space (X, d) into itself and let there exist $B \subset X$ such that

(1) $f(B) \subset B$

(2) for each $y \in B$ there is a positive integer $n(y)$ satisfying one of the following conditions :

(*a*) $d\left(f^{n(y)}(x), f^{n(y)}(y)\right) \leq \alpha d(x, y)$,

(*b*) $d\left(f^{n(y)}(x), f^{n(y)}(y)\right) \leq \beta\left[d\left(x, f^{n(y)}(x)\right) + d\left(y, f^{n(y)}(y)\right)\right]$,

(c) $d(f^{n(y)}(x), f^{n(y)}(y)) \leq \gamma [d(x, f^{n(y)}(y)) + d(y, f^{n(y)}(x))]$,

for all $x \in B$, $0 < \alpha < 1$, $0 < \beta < \frac{1}{2}$, $0 < \gamma < \frac{1}{4}$ and

(3) for some $x_0 \in B$, cl $\{f^n(x_0) : n \geq 1\} \subset B$, where cl denotes the closure.

Then there is a unique $u \in B$ satisfying $f(u) = u$ and $f^n(y_0) \to u$ for each $y_0 \in B$. Furthermore, if one of the following corresponding conditions

(i) $d(f^{n(u)}(x), f^{n(u)}(u)) \leq \alpha d(x, u)$

(ii) $d(f^{n(u)}(x), f^{n(u)}(u)) \leq \beta [d(x, f^{n(u)}(x)) + d(u, f^{n(u)}(u))]$

(iii) $d(f^{n(u)}(x), f^{n(u)}(u)) \leq \gamma [d(x, f^{n(u)}(u)) + d(u, f^{n(u)}(x))]$

is satisfied for all $x \in X$, then u is unique in X and $f^n(x_0) \to u$ for each $x_0 \in X$.

PROOF : It is to be noted that if we consider condition (a) of (2) only, then the theorem coincides with that of Guseman (1970). Therefore we shall discuss conditions (b) and (c) only.

If $y \in B$, then it is shown that $r(y) = \sup_n d(f^n(y), y)$ is finite (Khazanchi and Dass 1975). For $x_0 \in B$ as taken in (3), let $m_0 = n(x_0)$, $x_1 = f^{m_0}(x_0)$ and successively $m_i = n(x_i)$ and $x_{i+1} = f^{m_i}(x_i)$. By usual calculations we have for case (b) (refer Khazanchi and Dass 1975)

$$d(x_n, x_{n+1}) \leq \lambda^n (1 + \beta)^{n-1} d(x_0, f^{m_0}(x_0)) + \lambda^n \beta (1 + \beta)^{n-2} d(x_0, f^{m_1}(x_0))$$
$$+ \lambda^n \beta^2 (1 + \beta)^{n-3} d(x_0, f^{m_2}(x_0)) + \ldots + \lambda^n \beta^n d(x_0, f^{m_n}(x_0))$$

where

$$\lambda = \frac{\beta}{1 - \beta^2} \text{ giving}$$

$$d(x_n, x_{n+1} \leq [(\lambda^n \sum_{i=0}^{n-1} \beta^i (1 + \beta)^{n-i-1}) + \lambda^n \beta^n] r(x_0).$$

For $k > n$,

$$d(x_n, x_k) \leq \sum_{i=n}^{k-1} d(x_i, x_{i+1})$$
$$\leq \lambda^n [\sum_{i=0}^{n-1} \beta^i (1 + \beta)^{n-i-1}] [\sum_{i=0}^{k-n-1} \lambda^i (1 + \beta)^i] r(x_0)$$
$$+ [\sum_{i=n}^{k-2} \lambda^{i+1} \beta^i \sum_{j=0}^{k-i-2} \lambda^j (1 + \beta)^j] r(x_0)$$
$$+ [\lambda^n \beta^n \sum_{i=0}^{k-n-1} \lambda^i \beta^i] r(x_0)$$
$$< \frac{\lambda^n \sum_{i=0}^{n-1} \beta^i (1 + \beta)^{n-i-1}}{1 - \lambda(1 + \beta)} r(x_0) + \frac{\sum_{i=n}^{k-2} \lambda^{i+1} \beta^i}{1 - \lambda(1 + \beta)} r(x_0)$$
$$+ \frac{\lambda^n \beta^n}{1 - \lambda\beta} r(x_0)$$
$$\to 0 \text{ as } n, k \to \infty.$$

Therefore $\{x_n\}$ is a Cauchy sequence.

For case (c) (refer Khazanchi and Dass 1975)

$$d(x_n, x_{n+1}) \leq 2.3^{n-1} \delta^n d(f^{m_0}(x_0), x_0) + 2.3^{n-2} \delta^n d(f^{m_1}(x_0), x_0) + \ldots + 2.3^0 \delta^n d(f^{m_{n-1}}(x_0), x_0) + \delta^n d(x_0, f^{m_n}(x_0)),$$

$$\delta = \frac{\gamma}{1-\gamma}$$

$$\leq 2.3^{n-1} \delta^n \left[1 + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^{n-1}}\right] r(x_0) + \delta^n r(x_0),$$

$$< \frac{2.3^{n-1}\delta^n}{1-\frac{1}{3}} r(x_0) + \delta^n r(x_0)$$

$$= (3\delta)^n r(x_0) + \delta^n r(x_0)$$

and for $k > n$,

$$d(x_n, x_k) \leq \sum_{i=n}^{k-1} d(x_i, x_{i+1})$$

$$< (3\delta)^n [1 + 3\delta + (3\delta)^2 + \ldots + (3\delta)^{m-n-1}] r(x_0) + \delta^n [1 + \delta + \delta^2 + \ldots + \delta^{m-n-1}] r(x_0)$$

$$< \frac{(3\delta)^n}{1-3\delta} r(x_0) + \frac{\delta^n}{1-\delta} r(x_0)$$

$$\to 0 \quad \text{as} \quad n, k \to \infty,$$

showing thereby that $\{x_n\}$ is a Cauchy sequence in this case also.

Using completeness of X and (c) we get $x_n \to u \in B$. Thus there is an integer $n(u) \geq 1$ such that for case (b)

$$(d) \quad d(f^{n(u)}(y), f^{n(u)}(u)) \leq \beta [d(y, f^{n(u)}(y)) + d(u, f^{n(u)}(u))]$$

and for case (c)

$$(e) \quad d(f^{n(u)}(y), f^{n(u)}(u)) \leq \gamma [d(y, f^{n(u)}(u)) + d(u, f^{n(u)}(y))]$$

for each $y \in B$.

In case (b) we have

$$d(u, f^{n(u)}(u)) \leq d(u, x_n) + d(x_n, f^{n(u)}(x_n)) + d(f^{n(u)}(x_n), f^{n(u)}(u))$$
$$\leq d(u, x_n) + d(x_n, f^{n(u)}(x_n)) + \beta d(x_n, f^{n(u)}(x_n)) + \beta d(u, f^{n(u)}(u))$$

i.e.

$$d(u, f^{n(u)}(u)) \leq \frac{1}{1-\beta} d(u, x_n) + \frac{1+\beta}{1-\beta} d(x_n, f^{n(u)}(x_n))$$

Now,

$$\begin{aligned} d\left(x_n, f^{n(u)}(x_n)\right) &= d\left(f^{m_{n-1}}(x_{n-1}), f^{m_{n-1}} f^{n(u)}(x_{n-1})\right) \\ &\leq \beta d(x_{n-1}, x_n) + \beta d\left(f^{n(u)}(x_{n-1}), f^{n(u)}(x_n)\right) \\ &\leq \beta d(x_{n-1}, x_n) + \beta^2 d\left(x_{n-1}, f^{n(u)}(x_{n-1})\right) \\ &\quad + \beta^2 d\left(x_n, f^{n(u)}(x_n)\right) \end{aligned}$$

i.e.

$$\begin{aligned} d\left(x_n, f^{n(u)}(x_n)\right) &\leq \lambda d(x_{n-1}, x_n) + \lambda\beta d\left(x_{n-1}, f^{n(u)}(x_{n-1})\right), \ \lambda = \frac{\beta}{1-\beta^2} \\ &\leq \dots \\ &\leq \lambda d(x_{n-1}, x_n) + \lambda^2 \beta d(x_{n-2}, x_{n-1}) + \lambda^3\beta^2 d(x_{n-3}, x_{n-2}) \\ &\quad + \dots + \lambda^n \beta^n d\left(x_0, f^{n(u)}(x_0)\right) \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence

$$d\left(u, f^{n(u)}(u)\right) \to 0 \text{ as } n \to \infty.$$

Similarly in case (c) we have

$$\begin{aligned} d\left(u, f^{n(u)}(u)\right) &\leq d(u, x_n) + d\left(x_n, f^{n(u)}(x_n)\right) + d\left(f^{n(u)}(x_n), f^{n(u)}(u)\right) \\ &\leq d(u, x_n) + d\left(x_n, f^{n(u)}(x_n)\right) + \gamma d\left(x_n, f^{n(u)}(u)\right) \\ &\quad + \gamma d\left(u, f^{n(u)}(x_n)\right) \\ &\leq d(u, x_n) + d\left(x_n, f^{n(u)}(x_n)\right) + \gamma d(u, x_n) + \gamma d\left(u, f^{n(u)}(u)\right) \\ &\quad + \gamma d(u, x_n) + \gamma d\left(x_n, f^{n(u)}(x_n)\right) \end{aligned}$$

i.e.

$$d\left(u, f^{n(u)}(u)\right) \leq \frac{1+2\gamma}{1-\gamma} d(u, x_n) + \frac{1+\gamma}{1-\gamma} d\left(x_n, f^{n(u)}(x_n)\right)$$

However,

$$\begin{aligned} d\left(x_n, f^{n(u)}(x_n)\right) &= d\left(f^{m_{n-1}}(x_{n-1}), f^{m_{n-1}} f^{n(u)}(x_{n-1})\right) \\ &\leq \gamma d\left(x_{n-1}, f^{n(u)}(x_n)\right) + \gamma d\left(f^{n(u)}(x_{n-1}), x_n\right) \\ &\leq \gamma d(x_{n-1}, x_n) + \gamma d\left(x_n, f^{n(u)}(x_n)\right) \\ &\quad + \gamma d\left(x_{n-1}, f^{n(u)}(x_{n-1})\right) + \gamma d(x_{n-1}, x_n) \end{aligned}$$

i.e.

$$\begin{aligned} d\left(x_n, f^{n(u)}(x_n)\right) &\leq 2\delta\, d(x_{n-1}, x_n) + \delta d\left(x_{n-1}, f^{n(u)}(x_{n-1})\right), \ \delta = \frac{\gamma}{1-\gamma} \\ &\leq \dots \end{aligned}$$

$$\leq 2\delta d(x_{n-1}, x_n) + 2\delta^2 d(x_{n-2}, x_{n-1}) + \ldots$$
$$+ 2\delta^n d(x_0, x_1) + \delta^n d(x_0, f^{n(u)}(x_0))$$
$$\to 0 \text{ as } n \to \infty,$$

so that

$$d(u, f^{n(u)}(u)) \to 0 \text{ as } n \to \infty.$$

Hence

$$f^{n(u)}(u) = u.$$

Now (d) reduces to

$$d(f^{n(u)}(y), f^{n(u)}(u)) \leq \frac{\beta}{1-\beta} d(y, u)$$

while (e) reduces to

$$d(f^{n(u)}(y), f^{n(u)}(u)) \leq \frac{\gamma}{1-\gamma} d(y, u).$$

It now follows from the lemma provided by Guseman (1970) that u is the unique fixed point of f and $f^n(y_0) \to u$ for each $y_0 \in B$.

The last part of the theorem follows as a direct consequence of the discussion made above.

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