A THEOREM ON FIXED POINTS CONCERNING ITERATES OF A MAPPING

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The role of Banach's contraction principle can hardly be overemphasized in the development of various results related to fixed points. This principle has been generalized in various ways by different authors. Sehgal (1969) investigated mappings which were not necessarily contractions and established the existence of unique fixed point of a mapping having contractive iterates at each point of the space. In this paper, we have taken a set of conditions and have shown the existence of a unique fixed point in each case.

§1. The well-known Banach's contraction principle (Lusterink and Sobolov 1961) has played a significant role in various developmnets of the results related to fixed points. This result has been generalized by many authors, viz. Belluce and Kirk (1969), Dass and Gupta (1974), Edelstein (1961) and others. Kannan (1968), Singh (1970) and others have also made a study of fixed point theorems. Zamfirescu (1972) considered a set of three different conditions and established the existence of a unique fixed point if each of these conditions hold.

In the present paper, we discuss continuous self-mappings of a complete metric space satisfying a set of conditions and show that in each case such mappings possess a unique fixed point and the sequence of iterates of any point in the space converges to the fixed point. The proofs of the results follow the line of argument provided by Sehgal (1969).

§2. We prove the following.

Theorem—Let (X, d) be a complete metric space and $f: X \to X$ a continuous mapping such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$, at least one of the following conditions is satisfied :

$$d\left(f^{n(x)}(x), f^{n(x)}(y)\right) \leq \alpha d(x, y) \qquad \ldots(1)$$

$$d\left(f^{n(x)}(x), f^{n(x)}(y)\right) \leq \beta\left[d\left(x, f^{n(x)}(x)\right) + d\left(y, f^{n(x)}(y)\right)\right] \qquad \ldots(2)$$

$$d\left(f^{n(x)}(x), f^{n(x)}(y)\right) \leq \gamma\left[d\left(x, f^{n(x)}(y)\right) + d\left(y, f^{n(x)}(x)\right)\right] \qquad \ldots(3)$$

where

$$0 < \alpha < 1,\ 0 < \beta < \tfrac{1}{2} \ \text{and}\ 0 < \gamma < \tfrac{1}{2}.$$

Then f has a unique fixed point u and $f^n(x_0) \to u$ for each $x_0 \in X$.

In the above theorem when condition (1) holds, the result reduces to the one worked out by Sehgal. Therefore we shall discuss the theorem only when conditions (2) and (3) hold. We prove the following lemma first.

Lemma—If $f : X \to X$ be any mapping satisfying the conditions of the above theorem, then for each $x \in X$, $r(x) = \sup_n d(f^n(x), x)$ is finite.

PROOF : Let $x \in X$ and let

$$\eta(x) = \max.\ [d(f^s(x), x) : s = 1, 2, \ldots, n(x)]$$

If n is a positive integer, there exists an integer $t \geqslant 0$ such that $t\,n(x) < n \leq (t+1)\,n(x)$. We consider the conditions (2) and (3).

Condition (2)—We have

$$\begin{aligned} d(f^n(x), x) &\leq d(f^{n(x)} f^{n-n(x)}(x), f^{n(x)}(x)) + d(f^{n(x)}(x), x) \\ &\leq \beta d(f^{n-n(x)}(x), f^n(x)) + \beta d(x, f^{n(x)}(x)) + d(f^{n(x)}(x), x) \\ &\leq \beta d(f^{n-n(x)}(x), x) + \beta d(f^n(x), x) + (1+\beta)\,\eta(x) \end{aligned}$$

i.e.

$$\begin{aligned} d(f^n(x), x) &\leq \frac{1+\beta}{1-\beta}\eta(x) + \frac{\beta}{1-\beta} d(f^{n-n(x)}(x), x) \\ &\leq \frac{1+\beta}{1-\beta}\left[1 + \frac{\beta}{1-\beta} + \left(\frac{\beta}{1-\beta}\right)^2 + \ldots + \left(\frac{\beta}{1-\beta}\right)^t\right]\eta(x) \\ &< \frac{1+\beta}{1-\beta} \cdot \frac{1}{1 - \dfrac{\beta}{1-\beta}}\,\eta(x) \\ &= \frac{1+\beta}{1-2\beta}\,\eta(x) \text{ for all } n \geqslant 0. \end{aligned}$$

Hence $r(x) = \sup_n d(f^n(x), x)$ is finite.

Condition (3)—In this case

$$\begin{aligned} d(f^n(x), x) &\leq d(f^{n(x)} f^{n-n(x)}(x), f^{n(x)}(x)) + d(f^{n(x)}(x), x) \\ &\leq \gamma d(f^{n-n(x)}(x), f^{n(x)}(x)) + \gamma d(x, f^n(x)) + d(f^{n(x)}(x), x), \end{aligned}$$

i.e.

$$\begin{aligned} d(f^n(x), x) &\leq \frac{\gamma}{1-\gamma} d(f^{n-n(x)}(x), x) + \frac{1+\gamma}{1-\gamma}\eta(x), \\ &\leq \frac{1+\gamma}{1-\gamma}\left[1 + \frac{\gamma}{1-\gamma} + \left(\frac{\gamma}{1-\gamma}\right)^2 + \ldots + \left(\frac{\gamma}{1-\gamma}\right)^t\right]\eta(x) \\ &< \frac{1+\gamma}{1-\gamma} \cdot \frac{1}{1 - \dfrac{\gamma}{1-\gamma}}\,\eta(x) \\ &= \frac{1+\gamma}{1-2\gamma}\,\eta(x) \text{ for all } n \geq 0. \end{aligned}$$

Hence $r(x) = \sup_n d(f^n(x), x)$ is finite.

This proves the lemma.

Proof of the Theorem—Let $x_0 \in X$ be arbitrary.

Let

$$m_0 = n(x_0), \quad x_1 = f^{m_0}(x_0)$$

and successively

$$m_i = n(x_i), \quad x_{i+1} = f^{m_i}(x_i).$$

We show that $\{x_n\}$ is a convergent sequence.

Condition (2)—We have

$$\begin{aligned} d(x_1, x_2) &= d\left(f^{m_0}(x_0), f^{m_0} f^{m_1}(x_0)\right) \\ &\leq \beta d(x_0, x_1) + \beta d\left(f^{m_1}(x_0), f^{m_1}(x_1)\right) \\ &\leq \beta d(x_0, x_1) + \beta^2 d\left(x_0, f^{m_1}(x_0)\right) + \beta^2 d(x, x_2) \end{aligned}$$

i.e.

$$d(x_1, x_2) \leq \mu d(x_0, x_1) + \mu\beta d\left(x_0, f^{m_1}(x_0)\right),$$

where

$$\mu = \frac{\beta}{1 - \beta^2}.$$

Again,

$$\begin{aligned} d(x_2, x_3) &= d\left(f^{m_1}(x_1), f^{m_1} f^{m_2}(x_1)\right) \\ &\leq \beta d(x_1, x_2) + \beta d\left(f^{m}(x_1), f^{m_2}(x_2)\right) \\ &\leq \beta d(x_1, x_2) + \beta^2 d\left(x_1, f^{m_2}(x_1)\right) + \beta^2 d(x_2, x_3) \end{aligned}$$

i.e.

$$d(x_2, x_3) \leq \mu d(x_1, x_2) + \mu\beta d\left(x_1, f^{m_2}(x_1)\right).$$

However

$$\begin{aligned} d\left(x_1, f^{m_2}(x_1)\right) &= d\left(f^{m_0}(x_0), f^{m_0} f^{m_2}(x_0)\right) \\ &\leq \beta d(x_0, x_1) + \beta d\left(f^{m_2}(x_0), f^{m_2}(x_1)\right) \\ &\leq \beta d(x_0, x_1) + \beta^2 d\left(x_0, f^{m_2}(x_0)\right) + \beta^2 d\left(x_1, f^{m_2}(x_1)\right) \end{aligned}$$

i.e.

$$d\left(x_1, f^{m_2}(x_1)\right) \leq \mu d(x_0, x_1) + \mu\beta d\left(x_0, f^{m_2}(x_0)\right),$$

so that

$$\begin{aligned} d(x_2, x_3) &\leq \mu^2 d(x_0, x_1) + \mu^2 \beta d\left(x_0, f^{m_1}(x_0)\right) + \mu^2 \beta d(x_0, x_1) \\ &\quad + \mu^2 \beta^2 d\left(x_0, f^{m_2}(x_0)\right) \\ &\leq \mu^2 (1 + \beta) d(x_0, x_1) + \mu^2 \beta d\left(x_0, f^{m_1}(x_0)\right) \\ &\quad + \mu^2 \beta^2 d\left(x_0, f^{m_2}(x_0)\right). \end{aligned}$$

Similarly,

$$d(x_3, x_4) \leq \mu^3 (1+\beta)^2 d(x_0, x_1) + \mu^3 \beta (1+\beta) d(x_0, f^{m_1}(x_0)) + \mu^3 \beta^2 d(x_0, f^{m_2}(x_0)) + \mu^3 \beta^3 d(x_0, f^{m_3}(x_0))$$

and in general,

$$d(x_n, x_{n+1}) \leq \mu^n (1+\beta)^{n-1} d(x_0, f^{m_0}(x_0)) + \mu^n \beta (1+\beta)^{n-2} d(x_0, f^{m_1}(x_0)) + \mu^n \beta^2 (1+\beta)^{n-3} d(x_0, f^{m_2}(x_0)) + \ldots + \mu^n \beta^{n-1} \times (x_0, f^{m_{n-1}}(x_0)) + \mu^n \beta^n d(x_0, f^{m_n}(x_0)).$$

Now for $k > n$,

$$\begin{aligned} d(x_n, x_k) &\leq \sum_{i=n}^{k-1} d(x_i, x_{i+1}) \\ &\leq \mu^n \Big[\sum_{i=0}^{n-1} \beta^i (1+\beta)^{n-i-1}\Big]\Big[\sum_{i=0}^{k-n-1} \mu^i (1+\beta)^i\Big] r(x_0) \\ &\quad + \Big[\sum_{i=n}^{k-2} \mu^{i+1} \beta^i \sum_{j=0}^{k-i-2} \mu^j (1+\beta)^j\Big] r(x_0) \\ &\quad + \Big[\mu^n \beta^n \sum_{i=0}^{k-n-1} \mu^i \beta^i\Big] r(x_0) \\ &< \frac{\mu^n \sum_{i=0}^{n-1} \beta^i (1+\beta)^{n-i-1}}{1-\mu(1+\beta)} r(x_0) + \frac{\sum_{i=n}^{k-2} \mu^{i+1} \beta^i}{1-\mu(1+\beta)} r(x_0) \\ &\quad + \frac{\mu^n \beta^n}{1-\mu\beta} r(x_0) \\ &\to 0 \quad \text{as} \quad n, k \to \infty, \end{aligned}$$

showing thereby that $\{x_n\}$ is a Cauchy sequence.

Condition (3)—Here, we have

$$\begin{aligned} d(x_1, x_2) &= d(f^{m_0}(x_0), f^{m_0} f^{m_1}(x_0)) \\ &\leq \gamma d(x_0, x_2) + \gamma d(f^{m_1}(x_0), x_1) \\ &\leq \gamma d(x_0, x_1) + \gamma d(x_1, x_2) + \gamma d(f^{m_1}(x_0), x_0) + \gamma d(x_0, x_1) \end{aligned}$$

i.e.

$$d(x_1, x_2) \leq 2\delta d(x_0, x_1) + \delta d(f^{m_1}(x_0), x_0),$$

where

$$\delta = \frac{\gamma}{1-\gamma}.$$

Again,

$$\begin{aligned} d(x_2, x_3) &= d(f^{m_1}(x_1), f^{m_1} f^{m_2}(x_1)) \\ &\leq \gamma d(x_1, x_2) + \gamma d(x_2, x_3) + \gamma d(f^{m_2}(x_1), x_1) + \gamma d(x_1, x_2) \end{aligned}$$

i.e.

$$d(x_2, x_3) \leq 2\delta d(x_1, x_2) + \delta d\left(f^{m_2}(x_1), x_1\right).$$

As

$$d\left(f^{m_2}(x_1), x_1\right) = d\left(f^{m_0} f^{m_2}(x_0), f^{m_0}(x_0)\right)$$
$$\leq \gamma d\left(f^{m_2}(x_0), x_0\right) + \gamma d(x_0, x_1) + \gamma d(x_0, x_1) + \gamma d\left(x_1, f^{m_2}(x_1)\right)$$

i.e.

$$d\left(f^{m_2}(x_1), x_1\right) \leq 2\delta d(x_0, x_1) + \delta d\left(f^{m_2}(x_0), x_0\right),$$

therefore

$$d(x_2, x_3) \leq 4\delta^2 d(x_0, x_1) + 2\delta^2 d\left(f^{m_1}(x_0), x_0\right) + 2\delta^2 d(x_0, x_1) + \delta^2 d\left(f^{m_2}(x_0), x_0\right)$$
$$= 2.3^1 \delta^2 d(x_0, x_1) + 2.3^0 \delta^2 d\left(f^{m_1}(x_0), x_0\right) + \delta^2 d\left(f^{m_2}(x_0), x_0\right).$$

Similarly

$$d(x_3, x_4) \leq 2.3^2 \delta^3 d(x_0, x_1) + 2.3^1 \delta^3 d\left(f^{m_1}(x_0), x_0\right) + 2.3^0 \delta^3 d\left(f^{m_2}(x_0), x_0\right) + \delta^3 d\left(f^{m_3}(x_0), x_0\right)$$

and in general,

$$d(x_n, x_{n+1}) \leq 2.3^{n-1} \delta^n d\left(f^{m_0}(x_0)\, x_0\right) + 2.3^{n-2} \delta^n d\left(f^{m_1}(x_0), x_0\right) + 2.3^{n-3} \delta^n d\left(f^{m_2}(x_0), x_0\right) + \dots + 2.3^0 \delta^n d\left(f^{m_{n-1}}(x_0), x_0\right) + \delta^n d\left(x_0, f^{m_n}(x_0)\right)$$
$$\leq 2.3^{n-1} \delta^n \left[1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}}\right] r(x_0) + \delta^n r(x_0)$$
$$< \frac{2.3^{n-1} \delta^n}{1 - \frac{1}{3}} r(x_0) + \delta^n r(x_0)$$
$$= (3\delta)^n r(x_0) + \delta^n r(x_0).$$

Now for $m > n$,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1})$$
$$< (3\delta)^n [1 + 3\delta + (3\delta)^2 + \dots + (3\delta)^{m-n-1}]\, r(x_0) + \delta^n [1 + \delta + \delta^2 + \dots + \delta^{m-n-1}]\, r(x_0)$$
$$< \frac{(3\delta)^n}{1 - 3\delta} r(x_0) + \frac{\delta^n}{1 - \delta} r(x_0)$$
$$\to 0 \text{ as } n, m \to \infty,$$

showing that $\{x_n\}$ is a Cauchy sequence in this case also. We now show that $f(u) = u$.

Let $f(u) \neq u$. Then there exists a pair of disjoint closed neighbourhoods M and N such that $u \in M$, $f(u) \in N$ and

$$(*) \quad \eta = \inf. \ [d(x, y) : \ x \in M, y \in N] > 0.$$

Since f is continuous, $x_n \in M$ and $f(x_n) \in N$ for all large n.

For Condition (2), we have

$$\begin{aligned} d(f(x_n), x_n) &= d(f^{m_{n-1}} f(x_{n-1}), f^{m_{n-1}}(x_{n-1})) \\ &\leq \beta d(f(x_{n-1}), f(x_n)) + \beta d(x_{n-1}, x_n) \\ &\leq \beta d(f(x_{n-1}), x_{n-1}) + \beta d(x_{n-1}, x_n) + \beta d(f(x_n), x_n) \\ &\quad + \beta d(x_{n-1}, x_n) \end{aligned}$$

i.e.

$$\begin{aligned} d(f(x_n), x_n) &\leq \frac{\beta}{1-\beta} d(f(x_{n-1}), x_{n-1}) + \frac{2\beta}{1-\beta} d(x_{n-1}, x_n) \\ &\leq \left(\frac{\beta}{1-\beta}\right)^2 d(f(x_{n-2}), x_{n-2}) + \frac{2\beta^2}{(1-\beta)^2} d(x_{n-2}, x_{n-1}) \\ &\quad + \frac{2\beta}{1-\beta} d(x_{n-1}, x_n) \\ &\leq \dots\dots\dots\dots \\ &\leq \left(\frac{\beta}{1-\beta}\right)^n d(f(x_0), x_0) + 2\left(\frac{\beta}{1-\beta}\right)^n d(x_0, x_1) \\ &\quad + 2\left(\frac{\beta}{1-\beta}\right)^{n-1} d(x_1, x_2) + \dots + \frac{2\beta}{1-\beta} d(x_{n-1}, x_n) \\ &\to 0 \text{ as } n \to \infty, \text{ contradicting } (*). \end{aligned}$$

Similarly for condition (3)

$$\begin{aligned} d(f(x_n), x_n) &= d(f^{m_{n-1}} f(x_{n-1}), f^{m_{n-1}}(x_{n-1})) \\ &\leq \gamma d(f(x_{n-1}), x_n) + \gamma d(x_{n-1}, f(x_n)) \\ &\leq \gamma d(f(x_{n-1}), x_{n-1}) + \gamma d(x_{n-1}, x_n) + \gamma d(x_{n-1}, x_n) \\ &\quad + \gamma d(x_n, f(x_n)) \end{aligned}$$

i.e.

$$\begin{aligned} d(f(x_n), x_n) &\leq \delta d(f(x_{n-1}), x_{n-1}) + 2\delta\, d(x_{n-1}, x_n), \ \delta = \frac{\gamma}{1-\gamma} \\ &\leq \dots\dots\dots\dots \\ &\leq \delta^n d(f(x_0), x_0) + 2\delta^n d(x_0, x_1) + 2\delta^{n-1} d(x_1, x_2) + \dots \\ &\quad + 2\delta^2 d(x_{n-2}, x_{n-1}) + 2\delta d(x_{n-1}, x_n). \\ &\to 0 \text{ as } n \to \infty, \text{ contradicting } (*). \end{aligned}$$

Hence $f(u) = u$.

Uniqueness—Let there exist $v\,(\neq u) \in X$ such that $f(v) = v$. In condition (2)

$$\begin{aligned} d(u, v) &= d\big(f^{n(u)}(u), f^{n(u)}(v)\big) \\ &\leq \beta d\big(u, f^{n(u)}(u)\big) + \beta d\big(v, f^{n(u)}(v)\big) \\ &= 0 \end{aligned}$$

implying $u = v$.

Similarly in condition (3)

$$\begin{aligned} d(u, v) &= d\big(f^{n(u)}(u), f^{n(u)}(v)\big) \\ &\leq \gamma d\big(u, f^{n(u)}(v)\big) + \gamma d\big(v, f^{n(u)}(u)\big) \\ &\leq \gamma d(u, v) + \gamma d\big(v, f^{n(u)}(v)\big) + \gamma d(v, u) + \gamma d\big(u, f^{n(u)}(u)\big) \\ &= 2\gamma d(u, v) \end{aligned}$$

showing that $u = v$.

Lastly, we show that $f^n(x_0) \to u$.

Let $\eta^* = \max. \; [d\big(f^p(x_0), u\big) : \; p = 0, 1, 2, \ldots n(u) - 1]$.

If n is a sufficiently large integer, then $n = r.n(u) + q$, $0 \leqslant q < n(u)$, $r > 0$.

For condition (2)

$$\begin{aligned} d\big(f^n(x_0), u\big) &= d\big(f^{r.n(u)+q}(x_0), f^{n(u)}(u)\big) \\ &\leq \beta d\big(f^{(r-1)n(u)+q}(x_0), f^n(x_0)\big) + \beta d\big(u, f^{n(u)}(u)\big) \\ &\leq \beta d\big(f^{(r-1)n(u)+q}(x_0), u\big) + \beta d\big(f^n(x_0), u\big) \end{aligned}$$

i.e.

$$\begin{aligned} d\big(f^n(x_0), u\big) &\leq \frac{\beta}{1-\beta} d\big(f^{(r-1)n(u)+q}(x_0), u\big) \\ &\leq \ldots\ldots\ldots \\ &\leq \left(\frac{\beta}{1-\beta}\right)^r d\big(f^q(x_0), u\big) \\ &\leq \left(\frac{\beta}{1-\beta}\right)^r \eta^*. \end{aligned}$$

As $n \to \infty$ implies $r \to \infty$, we have

$$d\big(f^n(x_0), u\big) \to 0 \text{ as } n \to \infty.$$

For condition (3)

$$\begin{aligned} d\big(f^n(x_0), u\big) &= d\big(f^{r.n(u)+q}(x_0), f^{n(u)}(u)\big) \\ &\leq \gamma d\big(f^{(r-1)n(u)+q}(x_0), u\big) + \gamma d\big(u, f^n(x_0)\big) \end{aligned}$$

i.e.

$$d\left(f^{n}(x_0), u\right) \leq \frac{\gamma}{1-\gamma}\, d\left(f^{(r-1)\,n(u)+q}(x_0), u\right)$$
$$\leq \ldots\ldots\ldots$$
$$\leq \left(\frac{\gamma}{1-\gamma}\right)^{r} d\left(f^{q}(x_0), u\right)$$
$$\leq \left(\frac{\gamma}{1-\gamma}\right)^{r} \eta^{*}$$

As before $d\left(f^{n}(x_0), u\right) \to 0$ as $n \to \infty$.

This establishes the theorem completely.

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