ON SEMILATTICE RINGS

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In this paper we study the two radicals namely the Jacobson radical and the prime radical of a semilattice ring. We also find all the left divisors of zero and all right units of a semilattice ring.

§1. Let $(K, \leqslant, V)$ denote an upper semilattice where $(K, \leqslant)$ is a partially ordered set and for any two elements x, y in K, xVy denotes the least upper bound of x, y in K. Then K is a commutative semigroup of idempotents under the binary operation $x.y = xVy$ for all x, y in K. Let $R(K)$ denote the semigroupring (Clifford and Preston 1961, p. 158) of K over an arbitrary ring R. For any ring A let $H(A)$, $J(A)$ and $P(A)$ denote the hereditary radical property (Devinsky 1960, p. 125), Jacobson radical and the prime radical respectively of A. In this paper we prove that $H(R(K)) \subseteq H(R)(K)$, where by $H(R)(K)$ we mean the semigroupring of K over $H(R)$ considered as a ring. In fact we prove that $J(R(K)) = J(R)(K)$. Moreover if R has unity and K has a least element so that $R(K)$ has unity, we show that $P(R) \subseteq P(R(K)) \cap R$. But it has not been possible to prove the full converse of either of the inequalities $P(R(K)) \subseteq P(R)(K)$ or $P(R) \subseteq P(R)(K) \cap R$ even if $R(K)$ is assumed to have the identity element. However under the conditions that there are only a finite number of chains over each element of K and there is no chain of infinite length it has been possible to prove that $P(R(K)) \subseteq P(R)(K)$ and if besides satisfying above said conditions $R(K)$ has unity then $P(R) \subseteq P(R(K)) \cap R$. We also find all the left divisors of zero when R is integral domain (ring without divisors of zero). If $R(K)$ has unity then we also find all the right invertible elements of $R(K)$.

§2. Any element a in $R(K)$ has the form $a = \sum_{x \text{ in } k} r_x x$ where r_x in R is zero for all but a finite number of x in K. Define $a_y = \sum_{x \leqslant y} rx$ for each y in K and $|a| = \sum_{x \text{ in } k} r_x$. Denote the mapping $a \to a_y$ of $R(K)$ into itself by α_y and the mapping $a \to |a|$ of $R(K)$ into R by β. Then clearly β is an epimorphism. Denote the restriction of β to $\alpha_y(R(K))$ by β_y. Then we have,

Lemma 2.1—For each z in K, α_z is an endomorphism and β_z is an epimorphism.

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PROOF : Let $a = \sum_{x \text{ in } k} r_x x$, $b = \sum_{y \text{ in } k} s_y y$ belong to $R(K)$. Then clearly $(a+b)_z = a_z + b_z$ and $(ab)_z = \sum_{xy \leqslant z} r_x s_y\, xy = \sum_{x \leqslant z} r_x x \sum_{y \leqslant z} s_y y = a_z b_z$.

It follows in particular that $\alpha_z\left(R(K)\right)$ is a subring of $R(K)$. Hence β_z is a homomorphism which is obviously onto since $\beta_z(rz) = r$ for all r in R.

Corollary—$(aob)_z = a_z\, ob_z$ and $|(aob)_z| = |a_z|\, o\, |b_z|$.

Theorem 2.2—If H is a hereditary radical property then, $H\left(R(K)\right) \subseteq H(R)(K)$.

PROOF : Let $a = \sum_{i=1}^{n} r_i x_i$ be any element of $H\left(R(K)\right)$ where, x_i, $i = 1, 2, \ldots, n$ are so arranged that for $i < j$, $x_i \not> x_j$; $i, j = 1, 2, \ldots, n$. Let I be the principal ideal generated by a in $R(K)$. I is an H-ideal of $R(K)$ (Devinsky 1960, Theorem 48, p. 125). Now $(\beta\alpha_{x1})$ is an epimorphism of $R(K)$ on to R. Therefore $(\beta\alpha_{x_i})$ I is an H-ideal of R and $(\beta\alpha_{x_i})\, I$ contains r_1. Hence r_1 is in $H(R)$. Now assume (induction hypothesis) that $r_1, r_2, \ldots, r_{i-1}$ belong to $H(R)$, then $(\beta\alpha_{x_i})\, I$ contains $\sum_{x_j \leqslant x_i} r_j$. The set $\{j \mid x_j < x_i;\ j = 1, 2, \ldots, i-1\}$ may however be empty in which case $(\beta\alpha_{x_i})\, I$ contains r_i. Otherwise since $\sum_{j \leqslant i-1\, x_j \leqslant x_{i-1}} r_j$ is in $(\beta\alpha_{x_i})\, I$, hence r_i is in $(\beta\alpha_{x_i})\, I$. In both cases it follows that r_i is in $H(R)$.

Corollary—$J\left(R(K)\right) \subseteq J(R)(K)$ and $P\left(R(K)\right) \subseteq P(R)(K)$.

PROOF : Both J and P are hereditary radical properties.

Theorem 2.3—$J\left(R(K)\right) \supseteq J(R)(K)$.

PROOF: Let $a = \sum_{i=2}^{n} r_{x_i} x_i$ belong to $J(R)(K)$. Since a in $J(R)(K)$ implies ab in $J(R)(K)$ for all b in $R(K)$, to prove that a belongs to $J\left(R(K)\right)$ it suffices to show that a is right quasi-regular (r.q.r.). Let S be the subsemilattice of K generated by the subset $\{x_i \mid i = 1, 2, \ldots, n\} = T$. Then a can be written in the form $a = \sum_{x \text{ in } s} r_x x$, where $r_x = 0$ for all x not in T. Let x' be a minimal element of S. Then $r_{x'}\, os = 0$ for some s in $J(R)$ since r_x is in $J(R)$ for all x in S, so that $aosx' = \sum_{x \neq x'} r_x x - \sum_{x \neq x'} r_x s x x' = a'$ (say) where $a' = \sum_{y \text{ in } s'} r_y y$ belongs to $J(R)(S')$ and $S' = S \setminus \{x'\}$ is a subsemilattice of S. Since S is finite, repeating this process a finite number of times we get a right quasi-inverse (r.q.i.) of a in the form $sx'\, os'\, y'\, os''\, z'\, o \ldots.$

Corollary—$J\left(R(K)\right) = J(R)(K)$.

PROOF : Follows from corollary to Theorem 2.2 and Theorem 2.3.

Theorem 2.4—If $R(K)$ has unity then, $P(R) \supset P\left(R(K)\right) \cap R$.

PROOF: Since $R(K) \supseteq R, K$, it follows that any element of R which is strongly nilpotent (Lambek 1966, p. 55) in $R(K)$ is in particular strongly nilpotent in R.

Theorem 2.5—If K satisfies the conditions that there are only a finite number of chains over each element of K and every chain is of finite length then,

(i) $P(R)(K) \subseteq P(R(K))$.

(ii) If $R(K)$ also has unity then, $P(R) \subseteq P(R(K)) \cap R$.

PROOF: (i) Let $a = \sum_{i=1}^{n} r_{x_i} x_i$ belong to $P(R)(K)$, and let x_i's be so arranged that for $i < j, x_i \not> x_i$; $i, j = 1, 2, \ldots, n$. Then in the sequence, $a_0 = a$, $a_1 = a_0 b_0 a_0$, $a_2 = a_1 b_1 a_1, \ldots$, for arbitrary choice of $b_0, b_1, \ldots$ in $R(K)$, after a finite stage the coefficient r_{x_1} of x_1 vanishes, since r_{x_1} is in $P(R)$, so that we arrive at an element $a' = \sum_{i=1}^{m} s_{y_i} y_i, s_{y_i}$ in $P(R)$, $y_i \not> y_j$ for $i < j$; $i, j = 1, 2, \ldots, m$. Since K satisfies the conditions that there are only a finite number of chains over each of its elements and every chain is of finite length it follows by repeating the above process for a' as for a_0 and so on after a finite stage the sequence of elements $a_0, a', \ldots$. so obtained must vanish and hence the original sequence $a_0, a_1 \ldots$ must vanish, which implies that a is strongly nilpotent and hence in $P(R(K))$.

(ii) $R(K)$ having unity besides satisfying the above conditions implies that K is a finite semilattice with unity. Taking in particular for K the unity element in the left hand side of (i) we obtain (ii).

Theorem 2.6—Let S be a finite subsemilattice of K. Then there exists a unique element a in $R(K)$ such that if s_x, x in S are arbitrary elements of R, then,

(i) $|a_x| = s_x$, for all x in S.

(ii) Support $a \subset S$, where support of $a = \{x \text{ in } K \mid \text{coefficient of } x \text{ in } a \text{ is non-zero}\}$.

PROOF: S is obviously a finite set and the proof is by induction on the number of elements of S. If $S = \{x\}$ then, $a = s_x x$ satisfies (i) and (ii). Assume S has more than one element. Let x' be a minimal element of S. Then clearly $S \setminus \{x'\} = S'$ is a finite subsemilattice with one element less than S. By the induction hypothesis there exists an element a' in $R(K)$ with S replaced by S' and s_x replaced by s_z, z in S'. Taking $a = a' + s_x x'$, a thus defined satisfies conditions (i) and (ii) and is obviously unique, for if b is another such element of $R(K)$ satisfying (i) and (ii) then for $c = a - b$ in $R(K)$ we have $|c_y| = 0$ for all y in K by Lemma 2.1. Obviously this implies that $c = 0$, so that $a = b$.

Theorem 2.7—If R is an integral domain (ring without divisors of zero), then the set of all left divisors of zero of $R(K)$ is $Z = \{a \text{ in } R(K) \mid |a_x| = 0 \text{ for some } x \text{ in } K\}$.

PROOF: Let $0 \neq a$ in $R(K)$ be a left divisor of zero. Then there exists an element $0 \neq b$ in $R(K)$ such that $ab = 0$, which by Lemma 2.1 implies that $|a_x| \, |b_x| = 0$ for all x in K. Since $b \neq 0$ so that $|b_x| \neq 0$ for some x in K and R being on integral domain it follows that $|a_x| = 0$ so that a is in Z. Conversely assume that $0 \neq a = \sum_{i=1}^{n} r_{x_i} x_i$ is in Z such that $|a_{x_0}| = 0$ for some x_0 in K. Let S be the subsemilattice generated by the set $\{x_1, x_2, \ldots, x_n, x_0\}$. Define an element $b \neq 0$ such that $|b_{x_0}| = r \neq 0$, r in R, and $|b_x| = 0$ for all x in $S \setminus \{x_0\}$ and

$$|b_z| = \begin{cases} |b_y|, & \text{if } y = Vx_i, \ x_i \text{ in } S, \\ 0, & \text{if } z \geqslant x_i \text{ for any } x_i \text{ in } S. \end{cases}$$

Such a b exists and is uniquely determined by Theorem 2.6. We shall show that b is a right divisor of zero of a or in other words that $|(ab)_z| = 0$ for all z in K. If $x \geqslant x_0$, then $\| b_x| = 0$ so that $|(ab)_z| = 0$. If $x \geqslant x_0$, then there is a maximal element y is S such that $|a_x| = |a_y|, |b_x| = |b_y|$ and hence $|(ab)_z| = |a_x||b_x| = |a_y \| b_y| = 0$ since if $y = x_0$, then $|a_y| = 0$ and if $y \neq x_0$, $|b_y| = 0$.

Corollary 1—If for every x in K there exists a y in K such that $x > y$ or x is non-comparable to y, then every element of $R(K)$ is a divisor of zero.

Corollary 2—If K satisfies the descending chain condition (d.c.c.) for chains of elements of K and K has only a finite number of minimal elements then $R(K)$ contains non-zero divisors.

PROOF: The d.c.c. implies that there exists at least one minimal element in K. Let the set of all minimal elements of K be $\{x_1, x_2, \ldots, x_n\}$. Then the elements of the form $\sum_{i=1}^{n} r_{x_i} x_i$, r_{x_i} in R, $r_{x_i} \neq 0$ for $i = 1, 2, \ldots, n$; are all non-zero divisors in $R(K)$.

Theorem 2.8—If $R(K)$ has unity 1, when the set of all right units of $R(K)$ is $U = \{ a \text{ in } R(K) \mid |a_x| \text{ is a right unit for all } x \text{ in } K\}$.

PROOF: Let a be a right unit of $R(K)$, then there exists an element b in $R(K)$ such that $ab = 1$, so that, by Lemma 2.1, $|a_x \| b_x| = 1$ for all x in K. Conversely assume $a = \sum_{i=1}^{n} r_{x_i} x_i$ is in U such that for all z in K, $|a_z|$ is a right unit. The least element 1 in K is included in the set $\{x_i\}_{i=1}^{n}$, since otherwise $|a_1| = 0$. Let S be the finite subsemilattice generated by $\{x_i\}_{i=1}^{n}$. By Theorem 2.6 there exists an element b in $R(K)$ such that $|b_x| = s_x$ for all x in S, where $|a_x| s_x = 1$ and support $b = \{x \text{ in } K \mid \text{coefficient of } x \text{ in } b \text{ is non-zero}\} \subset S$. Clearly there does not exist any z in K such that $z \geqslant x_i$ for all x_i in S since $z \geqslant 1 = x_i$ for some i, $1 \leqslant i \leqslant n$. Hence $|(ab)_z| = |a_z \| b_z| = |a_y \| b_y| = 1$, where $y = \underset{z \geqslant x_i}{V} x_i$, x_i in S.

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