THE BLOCK-POINT TREE OF A GRAPH*

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The purpose of this paper is to introduce the notion of the block-point tree of a graph and characterize block-point trees. We also characterize those graphs whose block-point trees and subdivision graphs are isomorphic.

1. INTRODUCTION

All definitions not presented here may be found in Harary (1969). The graphs considered are finite undirected without loops or multiple lines. A 'cut-point' of a connected graph G is a point whose removal disconnects G. A 'block' of G is a maximal connected subgraph of G containing no cutpoints. The 'block-cutpoint tree' $bc(G)$ of a graph G is the graph whose points can be put in one-to-one correspondence with the set of blocks and cutpoints of G in such a way that two points of $bc(G)$ are adjacent if and only if one corresponds to a block B of G and the other to a cutpoint c of G and c is in B. The 'blockgraph' $B(G)$ of a graph G is the graph whose points are the blocks of G and in which two points are adjacent whenever the corresponding blocks have a cutpoint in common. The 'cutpoint graph' $C(G)$ of a graph G is the graph whose points are the cutpoints of G and in which two points are adjacent whenever the corresponding cutpoints lie on a common block of G.

We now define the 'block-point tree' $bp(G)$ of a connected graph G as the graph whose points can be put in one-to-one correspondence with the set of points and blocks of G in such a way that two points of $bp(G)$ are adjacent if and only if one corresponds to a block B of G and the other to a point v of G and v is in B. The definition of $bp(G)$ is a natural extension of the definition of the block-cutpoint tree $bc(G)$ and therefore, it is also an extension of the definitions of the block graph $B(G)$ and the cutpoint graph $C(G)$. In Fig. 1, a graph G and its block-point tree are shown. In $bp(G)$ the endpoints correspond to the noncutpoints of G.

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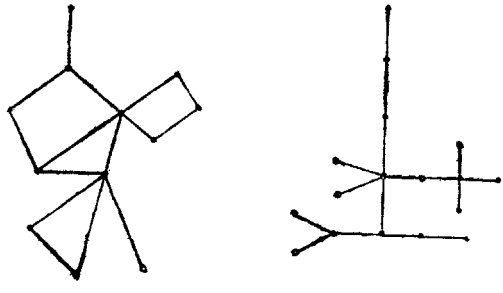

G $bp(G)$

FIG. 1

2. GRAPHS WITH BLOCK-POINT TREES

Before presenting a theorem due to Harary and Prins (1966) we note that if G_1 is the graph obtained from $bp(G)$ by deleting its endpoints, then $G_1 = bc(G)$.

Theorem—If G is connected, $bc(G)$ is a tree.

From the above note and theorem, we immediately conclude

Theorem 1—If G is connected, $bp(G)$ is a tree.

When defining any class of graphs, it is desirable to know the number of points and lines in each; this is easy to determine for block-point trees.

Theorem 2—If G is a connected graph with p points and b_i is the number of blocks to which point v_i belongs in G, then the block-point tree $bp(G)$ has $\sum_{i=1}^{p} b_i + 1$ points and $\sum_{i=1}^{p} b_i$ lines.

PROOF : It is shown (Harary 1959) that if G is connected with p points and b_i is the number of blocks to which point v_i belongs in G, then it has $\sum_{i=1}^{p} (b_i - 1) + 1$ blocks. Thus the number of points in $bp(G)$

$$= p + \sum_{i=1}^{p} (b_i - 1) + 1$$

$$= \sum_{i=1}^{p} b_i + 1.$$

Since $bp(G)$ is a tree, it has $\sum_{i=1}^{p} b_i$ lines.

Theorem 3—A connected graph G and its block-point tree $bp(G)$ are isomorphic if and only if G is trivial.

PROOF : Suppose $G = bp(G)$. If G is a nontrivial connected graph then G has at least one block. Thus the number of points of G is less than that in $bp(G)$ and hence $G \neq bp(G)$, a contradiction.

Conversely, if G is trivial, then clearly $G = bp(G)$.

A graph is 'bicolorable' if its points can be divided into two classes such that points of the same class are never adjacent. A graph is 'bicolored' if its points are divided into two classes such that points of the same class are never adjacent. A '*bc*-tree' is bicolored in which every endpoint has the same color. One can observe easily that in any *bc*—tree, the distance between any two endpoints is even.

Theorem 4—A graph G is the block-point tree of a connected graph if and only if it is a *bc*-tree.

PROOF : The block-point tree $bp(H)$ of a connected graph H is bicolorable by assigning one color to the points corresponding to the points of H and another color to the points corresponding to the blocks of G. Then clearly $bp(G)$ is bicolored and every endpoint of $bp(G)$ has the same color. Thus $bp(G)$ is a *bc*-tree.

Conversely, let G be a *bc*-tree. Construct a graph H corresponding to G such that $G = bp(H)$. Let G be colored blue and red with blue endpoints. We now construct a graph H whose points are the blue points of G and in which two points are adjacent whenever they are adjacent to the same red point in G. One can verify easily that $G = bp(H)$, completing the proof of the theorem.

Let H be the above constructed graph corresponding to G such that $G = bp(H)$. We note that if H_1 is another graph with $bp(H_1) = G$, then the number of lines of H_1 is less than or equal to the number of lines of H.

Theorem 5—If two connected graphs have the same block-point tree, then they have the same block graph.

PROOF : Construct a graph G_1 which has as points the blue points of $bp(G)$ and let two points be adjacent in G_1 if and only if they are adjacent to the same red point in $bp(G)$. It is easy to see that $B(G_1) = B(G)$.

The converse of Theorem 5 does not hold.

For example

$$B(K_2) = B(K_3),$$

but

$$bp(K_2) \neq bp(K_3).$$

Graphs with the same block-point tree also have the same cutpoint graph. But the converse does not hold.

Theorem 6—For any connected graph G, $bp(G) \neq bc(G)$.

PROOF : For trivial graph, the proof is trivial.

It is known that if G is a nontrivial connected graph, then G has at least two noncutpoints. Thus the number of points of $bc(G)$ is less than the number of points of $bp(G)$. Hence $bp(G) \neq bc(G)$, which completes the proof.

3. BLOCK POINT TREES AND SUBDIVISION GRAPHS

The ' square ' G^2 of a graph G is the graph whose points can be put in one-to-one correspondence with the set of points of G in such a way that two points of G^2 are adjacent if and only if the corresponding points of G are joined by a path of length one or two. The ' subdivision graph ' $S(G)$ of a graph G is the graph whose points can be put in one-to-one correspondence with the set of points and lines of G in such a way that two points of $S(G)$ are adjacent if and only if one corresponds to a line of G and the other to a point of G incident with it. It is known that the number of points of $S(G)$ is the sum of the number of points and lines of G. The ' total graph $T(G)$ ' of a graph G is the graph whose points can be put in one to one correspondence with the set of elements (i.e. the points and lines) of G in such a way that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (if both elements are points or both are lines) or incident (if one element is a point and the other a line). The purpose of this section is to characterize those graphs whose block-point trees and subdivision graphs are isomorphic.

Theorem 7—A connected graph G is a tree if and only if $bp(G) = S(G)$.

PROOF : Suppose G is a tree. Then every block of G is a line. Therefore, there is a one-to-one correspondence between the points of the block-point tree $bp(G)$ and the elements of G such that two points of $bp(G)$ are adjacent if and only if one corresponds to a line x of G and the other to a point of G incident with x. Then by the definition of $S(G)$, $bp(G) = S(G)$.

Conversely, suppose $bp(G) = S(G)$. If a connected graph G has a cycle, then the number of blocks of G is less than the number of lines of G. Thus, the number of points of $bp(G)$ is less than that in $S(G)$ and hence $bp(G) \neq S(G)$, a contradiction. This completes the proof of the theorem.

It is known (Behzad 1967) that for any graph G, the two graphs $T(G)$ and $(S(G))^2$ are isomorphic. Then by Theorem 7, we immediately conclude the following corollary.

Corollary 7.1—A connected graph G is a tree if and only if $T(G) = (bp(G))^2$.

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