THE SEMITOTAL-BLOCK GRAPH AND THE TOTAL-BLOCK GRAPH OF A GRAPH*

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In this paper we introduce the concepts of the semitotal-block graph and the total-block graph of a graph. We obtain some properties of these graphs and a characterization of semi-total-block graphs. We present a characterization of those graphs whose semitotal-block graphs and total-block graphs are isomorphic. We also characterize those graphs whose total-block graphs and total graphs are isomorphic.

1. INTRODUCTION

Definitions not given here may be found in Harary (1969). All graphs considered here are finite, undirected and without loops or multiple lines. The 'block point tree' $bp(G)$ of a graph G is the graph whose points can be put in one-to-one correspondence with the set of points and blocks of G in such a way that two points of $bp(G)$ are adjacent if and only if one corresponds to a block B of G and the other to a point v of G and v is in B. This concept was introduced by Kulli (1976).

If $B = \{u_1, u_2, \ldots, u_r;\ r \geqslant 2\}$ is a block of G, then we say that point u_1 and block B are 'incident' with each other, as are u_2 and B and so on. If two distinct blocks B_1 and B_2 are incident with a common cutpoint, then they are 'adjacent' blocks. The points and blocks of a graph are called its 'members'. We now define the 'semitotal-block graph' $T_b(G)$ of a graph G as the graph whose set of points is the union of the set of points and blocks of G and in which two points are adjacent if and only if the corresponding points of G are adjacent or the corresponding members are incident. We also define the 'total-block graph' $T_B(G)$ of a graph G as the graph whose points can be put in one-to-one correspondence with the set of points and blocks of G in such a way that two points of $T_B(G)$ are adjacent if and only if the corresponding members of G are adjacent or incident. In Fig. 1, a graph G, its semitotal-block graph $T_b(G)$ and its total-block graph

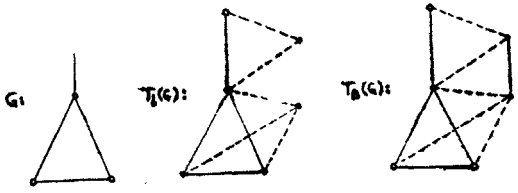

FIG. 1.

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$T_B(G)$ are shown. In $T_b(G)$ (or $T_B(G)$), the light points correspond to the points of G and the dark points correspond to the blocks of G. The graphs $T_b(G)$ and $T_B(G)$ have the same number of points. The graph $T_b(G)$ is a subgraph of $T_B(G)$. The block-point tree $bp(G)$ of G is a subgraph of $T_b(G)$ and consists all the points of $T_b(G)$ and dashed lines. The graph G is also a subgraph of $T_b(G)$. Clearly the graphs G and $bp(G)$ are both subgraphs of $T_B(G)$. The graph $T_B(G)$ contains both G and its block graph $B(G)$ as disjoint subgraphs.

2. Semitotal-Block Graphs

We start with a few preliminary results.

If v is a cutpoint of a graph G, then it is also a cutpoint of $T_b(G)$ and conversely.

A graph G and its semitotal-block graph $T_b(G)$ have the same number of cutpoints and blocks.

If G is a connected graph, then $T_b(G)$ is also a connected graph and conversely.

For any nontrivial complete graph K_p, $T_b(K_p) = K_{p+1}$.

The first theorem determines the number of points and lines in the semitotal-block graph of a graph.

Theorem 1—If G is a connected graph with p points and q lines and if b_i is the number of blocks to which point v_i belongs in G, then the semitotal-block graph $T_b(G)$ has $\sum_{i=1}^{p} b_i + 1$ points and $q + \sum_{i=1}^{p} b_i$ lines.

Proof : It is shown (Kulli 1976) that if G is a connected graph with p points and b_i is the number of blocks to which point v_i belongs in G, then $bp(G)$ has $\sum_{i=1}^{p} b_i + 1$ points and since the graphs $bp(G)$ and $T_b(G)$ have the same number of points, $T_b(G)$ has $\sum_{i=1}^{p} b_i + 1$ points.

The number of lines of $T_b(G)$ is the sum of the number of lines in G and in $bp(G)$. It is proved (Kulli 1976) that the number of lines in $bp(G)$ is $\sum_{i=1}^{p} b_i$. Therefore, the number of lines in $T_b(G)$ is $q + \sum_{i=1}^{p} b_i$.

Corollary 1.1—If G is a graph with p points, q lines and m components and if b_i is the number of blocks to which point v_i belongs in G, then the semitotal-block graph $T_b(G)$ has $\sum_{i=1}^{p} b_i + m$ points and $q + \sum_{i=1}^{p} b_i$ lines.

Theorem 2—A graph G is a totally disconnected graph if and only if the graphs G and $T_b(G)$ are isomorphic.

PROOF : The necessity is trivial.

For the sufficiency, it is easy to see that if G has a block, then the number of points of G is less than that in $T_b(G)$. Hence $G \neq T_b(G)$, a contradiction. This completes the proof.

Theorem 3—A graph G is a cycle if and only if the semitotal-block graph $T_b(G)$ is a wheel.

PROOF : The direct part of the theorem is trivial.

On the other hand, suppose $T_b(G)$ is a wheel with p points. Then $T_b(G)$ is a block and since the graphs G and $T_b(G)$ have the same number of blocks, G is a block. It is known that the wheel $T_b(G)$ has $2p-2$ lines and a point v of degree $p-1$. Thus $T_b(G) - v = G$ and it is a block with $p-1$ points and $p-1$ lines. It implies that G is a cycle, which completes the proof of the theorem.

Theorem 4—A graph G is the semitotal-block graph of some graph H if and only if

(i) every block B_i of G has a noncutpoint v_i such that G and $G - v_i$ have the same number of blocks

and

(ii) the degree of v_i is $n_i - 1$ where n_i is the number of points in B_i.

PROOF : Suppose $G = T_b(H)$, H some graph. Then clearly every block B_i of G has a noncutpoint v_i which corresponds to a block b_i of H and also the degree of v_i is $n_i - 1$ where n_i is the number of points in B_i. Since G and H have the same number of blocks, the graphs G and $G - v_i$ also have the same number of blocks.

Conversely, let (i) and (ii) be true for a graph G. Select a noncutpoint from each of the blocks of G such that each such point satisfies (i) and (ii). Let S denote the set of all such points. Let H be a graph obtained from G by removing the set of all points S. Clearly, H has as many blocks as G and each point of S corresponds to one block of H and is joined to all the points of this block. Hence $G = T_b(H)$.

3. TOTAL-BLOCK GRAPHS

If G is a connected graph, then $T_B(G)$ is also a connected graph and conversely.

For any nontrivial complete graph K_p, $T_B(K_p) = K_{p+1}$.

Theorem 5—If G is a connected graph with p points and q lines and if b_i is the number of blocks to which point v_i belongs in G, then the total-block graph $T_B(G)$ has $\sum_{i=1}^{p} b_i + 1$ points and $q + \sum_{i=1}^{p} \binom{b_i + 1}{2}$ lines.

PROOF : Since the graphs $T_b(G)$ and $T_B(G)$ have the same number of points and by Theorem 1, $T_B(G)$ has $\sum_{i=1}^{p} b_i + 1$ points.

The number of lines of $T_B(G)$ is the sum of the number of lines in $T_b(G)$ and in $B(G)$. By Theorem 1, the number of lines in $T_b(G)$ is $q + \sum_{i=1}^{p} b_i$. Also one easily verifies that $B(G)$ has $\sum_{i=1}^{p} \frac{b_i(b_i-1)}{2}$ lines. Hence the number of lines in $T_B(G)$

$$= q + \sum_{i=1}^{p} b_i + \sum_{i=1}^{p} \frac{b_i(b_i-1)}{2}$$

$$= q + \sum_{i=1}^{p} \binom{b_i+1}{2}.$$

Corollary 5.1—If G is a graph with p points, q lines and m components and if b_i is the number of blocks to which point v_i belongs in G, then the total block graph $T_B(G)$ has $\sum_{i=1}^{p} b_i + m$ points and $q + \sum_{i=1}^{p} \binom{b_i+1}{2}$ lines.

Theorem 6—No point of $T_B(G)$ is a cutpoint.

PROOF : Since $T_b(G)$ is a subgraph of $T_B(G)$, the only cutpoints of G may be the cutpoints of $T_B(G)$. Thus, it is sufficient to show that a cutpoint of a connected graph G is not a cutpoint of $T_B(G)$. Let v be a cutpoint of G. Then it lies in k blocks, say $B_1, B_2, \ldots, B_k, k \geqslant 2$. Let v_i be the corresponding point of B_i in $T_B(G)$ for $1 \leqslant i \leqslant k$. It is clear that $T_b(G) - v$ is disconnected and has k components say $G_1, G_2, \ldots, G_k$ such that $v_i \in G_i, 1 \leqslant i \leqslant k$. In $T_B(G)$, the induced subgraph with point set $\{v_1, \ldots, v_k\}$ is a complete graph. Thus $T_B(G) - v$ is connected. Hence v is not a cutpoint of $T_B(G)$, completing the proof.

Theorem 7—Let G be a connected graph. Then $T_b(G) = T_B(G)$ if and only if G is K_1 or a block.

PROOF : Suppose $T_b(G) = T_B(G)$. From Theorems 1 and 5 and the graphs $T_b(G)$ and $T_B(G)$ have $q + \sum_{i=1}^{p} b_i$ and $q + \sum_{i=1}^{p} \frac{b_i(b_i+1)}{2}$ lines respectively. Since $T_b(G)$ and $T_B(G)$ are isomorphic,

$$\sum_{i=1}^{p} b_i = \sum_{i=1}^{p} \frac{b_i(b_i+1)}{2}$$

or

$$\sum_{i=1}^{p} \frac{b_i(b_i-1)}{2} = 0.$$

It implies that $b_i = 0$ or $b_i = 1$.

If $b_i = 0$, then the number of blocks to which point v_i belongs in G is zero. Thus the degree of v_i in G is zero. Since G is connected, it is K_1.

If $b_i = 1$, then the number of blocks to which point v_i belongs in G is one. Thus v_i is not a cutpoint of G. Since G is connected, it is a block.

Conversely, if G is K_1 or a block, then clearly $T_b(G) = T_B(G)$.

One easily proves that a graph G is a cycle if and only if the total-block graph $T_B(G)$ is a wheel.

4. Total Graphs with Total-block Graphs

The total graph $T(G)$ of a graph G is the graph whose points can be put in one-to-one correspondence with the set of elements (i.e. the points and lines) of G in such a way that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (if both elements are points or both are lines) or incident (if one element is a point and the other a line). This concept was introduced by Behzad (1967). The *semitotal (point) graph* $T_2(G)$ of a graph G is the graph whose points can be put in one-to-one correspondence with the set of elements (i.e. the points and lines) of G in such a way that two points of $T_2(G)$ are adjacent if and only if the corresponding points of G are adjacent or the corresponding elements of G are incident (if one element is a point and the other a line). This concept was introduced by Sampathkumar and Chikkodimath (1973). We now characterize those graphs whose total-block graphs and total graphs are isomorphic.

Theorem 8—A connected graph G is a tree if and only if the graphs $T_B(G)$ and $T(G)$ are isomorphic.

Proof : Suppose G is a tree. Then $T_B(G) = T(G)$ since the lines and blocks of a tree coincide.

Conversely, suppose $T_B(G) = T(G)$, G is connected. If G has a cycle, then the number of blocks of G is less than the number of lines of G. It is known that the number of points of $T_B(G)$ is the sum of the number of points and blocks of G. Thus while $T(G)$ has $p + q$ points, $T_B(G)$ has less number of points. Hence $T_B(G) \neq T(G)$, a contradiction.

Theorem 9—A connected graph G is a tree if and only if the graphs $T_2(G)$ and $T_b(G)$ are isomorphic.

The proof is similar to the preceding ones and is omitted.

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