k-PARANORMAL OPERATORS

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Let H be a complex Hilbert space. An operator T on H is called k-paranormal if $\| Tx \|^k \leqslant \| T^k x \|$ $(k \geqslant 2)$ for $x \in H$, $\| x \| = 1$. The purpose of the present note is to study the following condition of k-paranormality of operators:

$$(*)\ T^{*k} T^k - 2z(T^*T)^{k/2} + z^2 \geq 0\ (z > 0).$$

The results obtained are the following :

(i) There exists k-paranormal operator, where $k > 2$, for which $(*)$ is not true.

(ii) A partial isometry satisfying $(*)$ is quasi-normal.

(iii) If T satisfies $(*)$ and if k an even integer different from 2, then $T^*x = \alpha^* x$ whenever $Tx = \alpha x$ and $\alpha \neq 0$.

Throughout the present note, T will denote an operator (a bounded linear transformation) on a complex Hilbert space H. Let $r(T)$ denote the spectral radius of T. T is defined to be quasi-normal if $T(T^*T) = (T^*T)T$; subnormal if it has a normal extension, and hyponormal if $T^*T \geq TT^*$. According to Istrăţescue (1967) T is paranormal or an operator of class (N) if $\| T^2 x \| \geq \| Tx \|^2$ for all unit vectors x in H. If $\| T^k x \| \geq \| Tx \|^k$ for all unit vectors x, then T is called k-paranormal or an operator of class $(N; k)$ (Istrăţescue and Istrăţescue 1967). By a normaloid operator, we mean an operator T such that $\| T^n \| = \| T \|^n$ for $n \geq 2$ or $r(T) = \| T \|$.

It is well known that every hyponormal operator is paranormal (Istrăţescue 1967) and every paranormal is k-paranormal (Istrăţescue *et al.* 1966). Also every k-paranormal is normaloid (Istrăţescue and Istrăţescue 1967).

Istrăţescue (1972, Theorem 1.3) has shown that if

$$T^{*k} T^k - 2z(T^*T)^{k/2} + z^2 \geq 0 \qquad (z > 0) \qquad \ldots(1)$$

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then T is k-paranormal. In fact, Andô (1972, Theorem 1) has shown that the condition (1) characterizes paranormal operators when $k = 2$. However, the next result shows that, when $k > 2$, it does not characterize k-paranormal operators.

Theorem 1—There exists a k-paranormal operator ($k > 2$) for which (1) is not true.

PROOF : Let K be the direct sum of a denumerable number of copies of H. Let A and B be two positive operators on H. Define an operator $T = T_{A,B,n}$ on K as follows :

$$T\langle x_1, \ldots x_n, x_{n+1}, x_{n+2}, \ldots\rangle = \langle 0, Ax_1, \ldots Ax_n, Bx_{n+1}, \ldots\rangle.$$

Let $n \geq k$. A simple computation shows that the operator T satisfies (1) if and only if

$$A^{k-m} B^{2m} A^{k-m} - 2zA^k + z^2 \geq 0 \, (z > 0), m = 1, 2, \ldots k - 1. \qquad \ldots(2)$$

To construct the required operator, we consider H to be a two-dimensional Hilbert space. Let $A = C^{1/2}$ and $B = (C^{-1/2} DC^{-1/2})^{1/2}$ be positive operators on H where

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}.$$

Then T is paranormal (Andô 1972, p. 172) and so it is k-paranormal for all $k \geq 2$. In particular, T is 4-paranormal. Since for $k = 4$, $m = 3$ and $z = 1$,

$$A^{k-m} B^{2m} A^{k-m} - 2zA^k + z^2 = \begin{bmatrix} 5 & 34 \\ 34 & 199 \end{bmatrix} \ngeq 0,$$

the desired conclusion follows.

We know that if a partial isometry is a hyponormal operator, then it turns out to be quasi-normal (Halmos 1967, problem 161). Therefore it is natural to inquire whether this result can be extended to k-paranormal operators. Here we are able to solve this problem when such operators satisfy the condition (1). As a particular case, it will follow that a paranormal partial isometry is quasi-normal.

Theorem 2—If a partial isometry T satisfies the condition (1), then it is quasi-normal.

PROOF : We know that an operator A is a partial isometry if and only if $AA^*A = A$ (Halmos 1967, Corollary 3 to problem 98). By our hypothesis,

$$T^{*k} T^k - 2z(T^*T) + z^2(T^*T)$$
$$= (T^*T)\{T^{*k}T^k - 2z(T^*T)^{k/2} + z^2\}(T^*T) \geq 0, \qquad (z > 0),$$

or

$$\langle (T^*T)x, x\rangle \leq \| T^k x \|^2,$$

for all x in H. Since T is a contraction, this inequality reduces to

$$\langle (T^*T)x, x\rangle \leq \| T^k x \|^2 \leq \| T^2 x \|^2 \leq \| Tx \|^2 = \langle T^*Tx, x\rangle$$

for all x in H, and hence $T^*T = T^{*2}T^2$.

Now following the argument of Embry (1973, Theorem 2), one can show that T is subnormal and hence quasi-normal.

It is well known that if T is hyponormal, then $Tx = \alpha x$ implies $T^*x = \alpha^* x$. Although the generalization of this result for k-paranormal operators remains as an unsolved problem, we have the partial solution in

Theorem 3—If T satisfies (1) and if k is an even positive integer such that $k \neq 2$, then $T^*x = \alpha^*x$ whenever $Tx = \alpha x$ and $\alpha \neq 0$.

PROOF : Let $k = 2n$ and assume $\| x \| = 1$. Then by (1) we have

$$| \alpha |^{4n} - 2z \langle (TT^*)^{n-1} Tx, Tx \rangle + z^2$$
$$= | \alpha |^{4n} - 2z | \alpha |^2 \langle (TT^*)^{n-1} x, x \rangle + z^2 \geq 0 \quad (z > 0).$$

This, in turn, implies

$$\langle (TT^*)^{n-1} x, x \rangle \leq | \alpha |^{2n-2}. \qquad \ldots(3)$$

Since we know that for a positive operator A and $1 \leqslant r < \infty$,

$$\langle A^r v, v \rangle \geq \langle Av, v \rangle^r, \; \| v \| = 1,$$

it follows from (3) that

$$\langle TT^* x, x \rangle^{n-1} \leq \langle (TT^*)^{n-1} x, x \rangle \leq | \alpha |^{2n-2}.$$

Consequently, $\| T^* x \| \leq | \alpha |$. Now applying this relation, one can easily prove $T^* x = \alpha^* x$.

Remark : The assertion of the above theorem is not necessarily true if $\alpha = 0$. To see this, let $T = T_{A,B,n}$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$A^{k-m} B^{2m} A^{k-m} - 2zA^k + z^2 = \begin{bmatrix} z^{2m-1} - 2z + z^2 & 0 \\ 0 & z^2 \end{bmatrix}$$

which is positive for $m = 1, 2, 3, \ldots, k-1$. In consequence, T satisfies (1) for all $k \geq 2$. However, if $y = \langle y_1, y_2, y_3 \ldots \rangle$, where $y_i = 0$ for $i \neq n+1$ and $y_{n+1} = x \in N(B)$ (the null space of B) be a vector in K, then $Ty = 0$ but $T^*y \neq 0$.

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