ON THE PROPAGATION OF MAGNETO-VISCOELASTIC WAVES IN A HALF-SPACE OF VOIGT TYPE OF MATERIAL

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This paper deals with the study of the propagation of magneto-viscoelastic plane waves in a half-space of Voigt type of material. The equations of electromagnetism and of mechanics of continua have been made use of in the present study. The different characteristics of propagation with accompanying effects of magnetic fields have been discussed.

1. INTRODUCTION

The propagation of waves in an elastic medium is, doubtless, an interesting problem, (Kolsky 1963) and the study becomes more interesting if the medium be viscoelastic. Such studies acquire more complexity if the media are acted upon by a magnetic field. Although a number of studies have been undertaken in recent years Kaliski (1961), Sinha (1965) Paria (1962), Willson (1963), Bakshi (1969), Ray Chaudhuri (1971) and Nowacki (1962) regarding the propagation of distrubances in the elastic media in the presence of magnetic and thermal fields, but no attempt has yet been made to investigate fully the propagation of such waves in a viscoelastic medium which is permeated by a magnetic field. Of course, the recent paper of Basu Mallick (1970) considers the problem of the propagation of waves in a visco-elastic medium in contact with a fluid medium acted upon by a magnetic field. This paper reports not merely the frequency equation but it aims at discussing the striking and characteristic features of propagation of waves in the media. The group velocities for certain ranges of frequencies have also been discussed conclusively.

2. STATEMENT OF THE PROBLEM, AND FUNDAMENTAL EQUATIONS

We consider a finitely conducting viscoelastic material of Voigt type in the half-space $z \geqslant 0$ (the half space $z < 0$ being electromagnetically insulated), the z-axis being directed normally into the medium. The medium is excited by a given primary magnetic field. Our object is to investigate the propagation of magneto-viscoelastic waves in the medium. The problem being one of inter-action between mechanical and electromagnetic fields, the fundamental equations of the problem are evidently those of viscoelasticity and the electromagnetic equations of Maxwell.

The equations of Maxwell for the problem at hand are given by (see Paria 1962) :

$$\left.\begin{array}{ll} \operatorname{curl} \vec{H} = \vec{J}, & \operatorname{curl} \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \operatorname{div} \vec{B} = 0, & \vec{B} = \mu_e \vec{H}, \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Here $\vec{E}$ and $\vec{H}$ are, respectively, the electric and magnetic intensity vectors, $\vec{J}$ the current density vector, μ_e the permeab'lity of the medium and $\vec{B}$ the magnetic induction. The displacement current has been neglected on account of the fact, that, in magnetoelastic disturbances, such currents do not have appreciable effects (see Kaliski 1961).

The generalized Ohm's law in the deformable medium is

$$\vec{J} = \sigma \left[\vec{E} + \left(\frac{\partial \vec{u}}{\partial t} \times \vec{B} \right) \right] \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where $\vec{u}$ is the displacement vector in the strained solid and σ is the electrical conductivity of the medium. Here the scalar σ is assumed for reasons of simplicity, to be uniform and homogeneous throughout the elastic medium. This assumption is also borne out by the experimental considerations, vide, Kaliski (1961).

The equations of motion are given by (see Paria 1962) :

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\delta \tau_{ij}}{\partial x_k} + (\vec{J} \times \vec{B})_i, \; i, j = 1, 2, 3 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where

$$\tau_{ij} = 2\mu e_{ij} + \lambda \, e \, \delta_{ij} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

and

$$e = div \; \vec{u} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

With the aid of the expressions (4) to (6) we can write eqns. (3) in the form

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \operatorname{grad} e + (\vec{J} \times \vec{B}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2} . \quad \ldots \quad \ldots \quad (7)$$

We are considering in our problem the viscoelastic material to be of Voigt type; we thus take

$$\lambda = \lambda_0 + \lambda_1 \frac{\partial}{\partial t}, \mu_0 = \mu + \mu_1 \frac{\partial}{\partial t} \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

where λ_0, μ_0, λ_1 and μ_1 are material constants. Now the equation of motion (7) with the help of (8) takes the form

$$\left(\mu_0 + \mu_1 \frac{\partial}{\partial t} \right) \nabla^2 \vec{u} + \left\{ (\lambda_0 + \mu_0) + (\lambda_1 + \mu_1) \frac{\partial}{\partial t} \right\} \text{grad } e + (\vec{J} \times \vec{B})$$

$$= \rho \frac{\partial^2 \vec{u}}{\partial t^2}. \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

One way of studying the problem from a qualitative standpoint would be to seek unidirectional investigations as in Kaliski (1961) and Paria (1962).

The magnetoelastic disturbances are set into the medium by applying the uniform magnetic field of intensity $\vec{H}$. The application of such a field would, according to principles of electromechanical interactions, vide Kaliski (1961), sets up a mechanical field in the transverse direction. We choose it in the direction of z-axis, as in Paria (1962).

Let us seek one-dimensional disturbances in the z direction only; all vectors are taken to be functions of z and t only and independent of x and y coordinates. We assume in this case that the displacement component vector $\vec{u}$ and magnetic field vector $\vec{H}$ have components parallel to the co-ordinate axes (o, o, u) and (H, O, O) respectively. With the above assumptions and following the procedure of Paria (1962), we obtain, from eqns. (1), (2), and (9) after simplifications the two fundamental equations of the problem, viz.

$$\frac{\partial h_x}{\partial t} = \nu_H \frac{\partial^2 h_x}{\partial z^2} - H \frac{\partial^2 u}{\partial z \partial t} \qquad \ldots \quad \ldots \quad \ldots \quad (10)$$

and

$$\left(c_1^2 + k^2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial z^2} - \frac{\mu_e H}{\rho} \frac{\partial h_x}{\partial z} = \frac{\partial^2 u}{\partial t^2} \qquad \ldots \quad \ldots \quad (11)$$

where $\nu_H = (\mu_e \sigma)^{-1}$ is the magnetic viscosity, h_x^1 the perturbation of the component of the magnetic field along the x-axis and $c_1 = \left(\frac{\lambda_0 + 2\mu_0}{\rho} \right)^{1/2}$ and $k = \left(\frac{\lambda_1 + 2\mu_1}{\rho} \right)^{1/2}$.

For a perfect conductor, the conclusion remains the same as in the elastic case. For a conductor characterised by a finite conductivity, let us take the solution of eqns. (10) and (11) in the forms

$$\left\{ \begin{matrix} h_x \\ u \end{matrix} \right\} = \left\{ \begin{matrix} h'_x \\ u' \end{matrix} \right\} \exp\left[i\,(qz - \omega t)\right] \quad \ldots \quad \ldots \quad (12)$$

where h'_x and u' are two arbitrary constants.

Substituting the expressions (12) in equations (10) and (11), we get two equations for u' and h'_x and these lead to the dispersion relation

$$\omega^3 + i\omega^2\,(\nu_H + k^2)\,q^2 - \omega q^2 \left(c_1^2 + \nu_H q^2 k^2 + \frac{\mu_e H^2}{\rho} \right) - i\nu_H\, c_1^2\, q^4 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

This is a cubic equation in ω. Equation (13) shows that although the magnetic field aligns itself with the elastic part of the disturbances, there appears also a term exhibiting elastic behaviour only. Let us now analyse the above frequency equation (13) for some particular cases.

Case I : Let the wavelength 'L' $= \dfrac{2\pi}{q}$ *be real i.e., 'q' be real*

(i) When the wavelength 'L' is large i.e. when q is small, then the frequency equation (13) reduces to the form

$$\omega^3 + i\omega^2(\nu_H + k^2)q^2 - \omega q^2 c_1^2\,(1 + R_H) = 0 \quad \ldots \quad \ldots \quad (14)$$

where $R_H = \dfrac{\mu_e H^2}{\rho c_1^2}$ is the magnetic pressure number, vide, Paria (1962).

Solving (14), we get

$$\omega = 0, \quad (\pm X - i\,Y) \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where

$$X = qc_1\sqrt{1 + R_H}$$

$$Y = \frac{q^2}{2}(\nu_H + k^2).$$

Therefore from (12) with the aid of the results (15), we get

$$u = u' \exp(iqz) + \exp(-Yt)\,\{u'' \exp[i(qz - Xt) + u''' \exp[i(qz + Xt)]\} \quad (16)$$

where u'' and u''' are two arbitrary constants.

This represents two waves, one propagating forward and the other backward, but amplitudes of each of them would decrease exponentially in time with exponent inversely proportional to the square of the wavelength. But in this case the effect of the magnetic field is conspicuous.

(ii) When the wavelength L is small i.e. when $1/q$ is small; then from equation (13) neglecting the higher powers of $1/q$, we get

$$\omega = -\frac{ic_1^2}{k^2} \quad .. \quad .. \quad .. \quad (17)$$

Therefore from (12) with the aid of (17), we get

$$u = u' \exp\left[-\frac{c_1^2}{k^2}t\right] \exp\ (iqz) \quad .. \quad .. \quad .. \quad (18)$$

From the nature of the expression of u given by (18), we observe that the magneto-viscoelastic waves are damped out in time as it is evident from the term $\exp\left(-\frac{c_1^2}{k^2}t\right)$. Further, we note that there is no effect of magnetic field for small wavelength in the damping factor and also that the wave character is destroyed.

Case 2 : Let the frequency ω be real

From the frequency equation (13), we get

$$\nu_H c_1^2(1 - ik_1\omega)q^4 - \omega\left\{\omega(\nu_H + k^2) + i\left(c_1^2 + \frac{\mu_e H^2}{\rho}\right)\right\} q^2 - i\omega^3 = 0 \quad (19)$$

where

$$k_1 = \frac{k^2}{c_1^2}$$

(i) When ω is small, then from the expression (19), we get

$$q = 0, \qquad \pm(X_1 - iY_1) \quad .. \quad .. \quad (20)$$

where $$X_1 = Y_1 = \sqrt{\frac{\omega}{2\nu_H}(1+R_H)}. \quad .. \quad .. \quad .. \quad (21)$$

Now the phase velocity C' is given by

$$C' = \frac{\omega}{X_1} = \frac{(2\nu_H)^{\frac{1}{2}}\ \omega^{\frac{1}{2}}}{\sqrt{(1+R_H)}} \quad .. \quad .. \quad (22)$$

and the group velocity C_g is given by the standard formula, viz.

$$C_g = C'\left[1 - \frac{\omega}{C'}\frac{dC'}{d\omega}\right]^{-1} = \frac{2(2\nu_H)^{\frac{1}{2}}\omega^{\frac{1}{2}}}{\sqrt{(1+R_H)}} \quad \ldots \quad (23)$$

From the expressions (22) and (23) we can conclude that group velocity is twice the phase velocity. Both the phase and group velocities of the waves depend on the frequency and also on the intensity of the magnetic field.

(ii) When ω is large, then from eqn. (19), we get

$$q = \pm(X_2 + iY_2) \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where

$$X_2 = \frac{1}{\sqrt{2}}\left[1 - \frac{c_1^2(1+R_H)}{\omega(\nu_H+k^2)}\right] \text{ and } Y_2 = \frac{1}{\sqrt{2}}\left[1 + \frac{c_1^2(1+R_H)}{\omega(\nu_H+k^2)}\right]$$

Therefore the phase velocity C' is given by

$$C' = \frac{\omega}{X_2} = \omega\left[1 + \frac{c_1^2(1+R_H)}{\omega\ (\nu_H+k^2)}\right] \quad \ldots \quad \ldots \quad (25)$$

and the group velocity C_g is obtained as

$$C_g = \omega\left[1 + \frac{(\nu_H+k^2)\omega}{c_1^2(1+R_H)}\right]. \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

From the expressions (25) and (26), it is evident that when the frequency is large, both the group and phase velocities become infinite. Although this appears to be physically absurd, one might interpret this as a wave packet consisting of infinitesimally short wavelengths propagating with an infinite velocity and as such, it may be insignificant physically.

The results and discussions arrived at the foregoing sections have relevance to problem on disturbances in the interior of the earth, taking into account the viscoelastic nature of the materials. It is believed such analyses would be of use in detection of mechanical explosions inside the earth making use of magneto-elastic disturbances, as indicated by Kaliski (1961), Cagniard (1954), Rikitake (1952) and Yukutake (1967). In particular it seems that the group velocity would determine the electromechanical energy developed in the interior of the earth.

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