TRANSFORMATIONS OF THE FUNCTION OF SEVERAL VARIABLES

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(Communicated by F. C. Auluck, F.N.A.)

(*Received 26 October 1974*)

The object of this paper is to establish a transformation formula for generalized hypergeometric function of several variables. Few particular interesting cases have also been discussed.

1. INTRODUCTION

Recently Srivastava [1970, p. 1079, (1)] defines the hypergeometric function of several variables by means of the following equality:

$$f(z_1, \ldots, z_r) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} C(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{z_i^{k_i}}{k_i!}, \quad \ldots(1.1)$$

where coefficients $C(k_1, \ldots, k_r)$, $k_i \geq 0$, $1 \leqq i \leqq r$, are arbitrary constants subject to the appropriate conditions of (absolute) convergence so that the two sides are meaningful in every case.

The object of this paper is to establish a transformation formula for the function of several variables defined by (1.1). Obviously, due to the generalized nature of the function involved in the formula, the result obtained here, provides us with the known results in one and two arguments. By the proper choice of the parameters few interesting particular cases have been obtained, some of them are already known while some others are believed to be seen for the first time in the literature.

2. FORMULA

We shall establish the formula :

$$(1 - \Sigma Z_r)^{-\lambda} f\left[\left(\frac{-z_1}{1 - \Sigma Z_r}\right)^{\delta}, \ldots, \left(\frac{-z_r}{1 - \Sigma Z_r}\right)^{\delta}\right]$$

$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{(\lambda, N)}{n_1! \ldots n_r!} z_1^{n_1} \ldots z_r^{n_r}$$

$$\times \sum_{k_1=0}^{[n_1/\delta]} \cdots \sum_{k_r=0}^{[n_r/\delta]} C(k_1, \ldots, k_r) \frac{\prod_{i=1}^{r} [(\Delta(\delta, -n_i), k_i)]}{(\Delta(\delta, \lambda), K)\, k_1! \ldots k_r!}, \quad \ldots(2.1)$$

where $|\Sigma Z_{r-1}| + |z_r| < 1$, r is a positive integer and $|z_i/1 - \Sigma Z_r|$, $i = 1, 2, \ldots, r$ are constrained appropriately. The function f on the left-hand side of (2.1) is given by (1.1) subject to such conditions so that the series involved is either (absolutely) convergent or terminating. For convenience the abbreviation $\Delta(\delta, \alpha)$ has been used to denote the set of parameters

$$\frac{\alpha}{\delta}, \frac{\alpha+1}{\delta}, \ldots, \frac{\alpha+\delta-1}{\delta}; \quad \delta = 1, 2, \ldots\ldots$$

and

$$(\alpha, n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1 & , \text{ if } n = 0 \\ \alpha(\alpha+1)\ldots(\alpha+\delta-1), & \text{if } n = 1, 2, \ldots \end{cases} \qquad \ldots(2.2)$$

For the sake of brevity, we have used

$$N = n_1 + \ldots + n_r$$

$$K = k_1 + \ldots + k_r$$

and

$$\Sigma Z_r = z_1 + \ldots + z_r.$$

This interpretation shall be retained throughout the paper.

In proving (2.1), we shall use the following multinomial theorem

$$(1 - \Sigma Z_r)^{-\lambda} = \sum_{m_1=0}^{\infty} \ldots \sum_{m_r=0}^{\infty} (\lambda, M) \frac{z_1^{m_1}}{m_1!} \ldots \frac{z_r^{m_r}}{m_r!}, \qquad \ldots(2.3)$$

where $|\Sigma Z_{r-1}| + |z_r| < 1$, and as usual $M = m_1 + \ldots + m_r$.

It can be proved with the help of mathematical induction [Mostow *et al.* 1963, p, 43 (1)]. We use induction on r. We denote (2.3) as the statement P_r and let there be assigned to every positive integer r, a statement P_r which may be either true or false. Now P_1 is true since for $r = 1$ we obtain

$$(1 - z_1)^{-\lambda} = \sum_{m_1=0}^{\infty} \frac{(\lambda, m_1)}{m_1!} z_1^{m_1}; |z_1| < 1, \qquad \ldots(2.4)$$

which is well known binomial theorem. Now suppose P_r is true for some r; we show P_{r+1} must also be true, we have

$$(1 - \Sigma Z_{r+1})^{-\lambda} = (1 - \Sigma Z_r)^{-\lambda} \left(1 - \frac{z_{r+1}}{1 - \Sigma Z_r}\right)^{-\lambda}.$$

Since

$$|\Sigma Z_r| + |z_{r+1}| < 1 \Rightarrow \left|\frac{z_{r+1}}{1 - \Sigma Z_r}\right| < 1.$$

Hence using (2.4), we obtain

$$(1-\Sigma Z_{r+1})^{-\lambda}=\sum_{m_{r+1}=0}^{\infty}\frac{(\lambda, m_{r+1})}{m_{r+1}\,!}\,z_{r+1}^{m_{r+1}}\,(1-\Sigma Z_r)^{-(\lambda+m_{r+1})}.$$

Since P_r is true, we have

$$(1-\Sigma Z_{r+1})^{-\lambda}=\sum_{m_{r+1}=0}^{\infty}\frac{(\lambda, m_{r+1})}{m_{r+1}\,!}\,z_{r+1}^{m_{r+1}}$$

$$\times\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(\lambda+m_{r+1}, M)}{m_1\,!\ldots m_r\,!}\,z_1^{m_1}\ldots z_r^{m_r},$$

$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_{r+1}=0}^{\infty}\frac{(\lambda, m_1+\ldots+m_{r+1})}{m_1\,!\ldots m_{r+1}\,!}\,z_1^{m_1}\ldots z_{r+1}^{m_{r+1}}.$$

Therefore P_{r+1} is true if P_r is true. Since P_1 is true as noted above; all P_r are true, i.e. (2.3) holds for all positive integer r.

Now we proceed to prove our main formula (2.1).

PROOF of (2.1) : In order to prove (2.1), we start from its left-hand side Ω say; substitute from (1.1) to obtain

$$\Omega=\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}C(k_1,\ldots,k_r)\prod_{i=1}^{r}\left[\frac{(-z_i)^{\delta k_i}}{k_i\,!}\right](1-\Sigma Z_r)^{-(\lambda+\delta K)}.$$

Here using (2.3) and inverting the order of inner and outer summations, since it is justified due to absolute convergence of the series involved in the process, we get

$$\Omega=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}C(k_1,\ldots,k_r)\frac{(-)^{\delta K}(\lambda+\delta K, M)}{k_1,!\ldots k_r!}$$

$$\times\frac{z_1^{\delta k_1+m_1}}{m_1\,!}\cdots\frac{z_r^{\delta k_r+m_r}}{m_r\,!}.$$

Now substituting $\delta k_i+m_i=n_i$ and making use of the elementary relationships (Rainville 1963, p. 22, 32)

$$(\alpha,\delta k)=\delta^{\delta k}\prod_{i=1}^{\delta}\left[\left(\frac{\alpha+i-1}{\delta},k\right)\right]$$

and

$$(-n, k) = \frac{n\,!\,(-)^k}{(n-k)\,!}, \qquad \ldots(2.5)$$

the required result (2.1) is obtained after a little simplification.

Our formula (2·1) is capable of giving us a number of transformation formulae, whenever we can sum its right-hand side by choosing $C(k_1, \ldots, k_r)$ suitably.

To facilitate the discussion, we substitute

$$C(k_1, \ldots, k_r) = \frac{\prod_{i=1}^{A}(a_i, K) \prod_{i=1}^{B_1}(b_i^{(1)}, k_1) \ldots \prod_{i=1}^{B_r}(b_i^{(r)}, k_r)}{\prod_{i=1}^{C}(c_i, K) \prod_{i=1}^{D_1}(d_i^{(1)}, k_1) \ldots \prod_{i=1}^{D_r}(d_i^{(r)}, k_r)},$$

so that (2.1) assumes the form

$$(1 - \Sigma Z_r)^{-\lambda}\; F\left[\begin{matrix}(a_A) : (b_{B_1}^{(1)}); \ldots (b_{B_r}^{(r)}) ; \\ (c_C) : (d_{D_1}^{(1)}) \ldots (d_{D_r}^{(r)}) ;\end{matrix} \left(\frac{-z_1}{1-\Sigma Z_r}\right)^{\delta}, \ldots, \left(\frac{-z_r}{1-\Sigma Z_r}\right)^{\delta}\right]$$

$$= \sum_{n_1=0}^{\infty} \ldots \sum_{n_r=0}^{\infty} \frac{(\lambda, N)}{n_1 \ldots n_r\,!} z_1^{n_1} \ldots z_r^{n_r}$$

$$\times F\left[\begin{matrix}(a_A) & : (b_{B_1}^{(1)}), \Delta(\delta, -n_1) ; \ldots ; (b_{B_r}^{(r)}), \Delta(\delta, -n_r); \\ (c_C), \Delta(\delta, \lambda) & : (d_{D_1}^{(1)}) \qquad ; \ldots ; (d_{D_r}^{(r)}) \qquad ;\end{matrix} 1, \ldots, 1\right]$$

$$\ldots(2.6)$$

where $|\Sigma Z_{r-1}| + |z_r| < 1$, r is any positive integer; $A + B_i \leq C + D_i$ but if $A + B_i = C + D_i + 1$ then $|Z_i/(1 - \Sigma Z_r)|$ are constrained appropriately, (a_A) is taken to abbreviate the sequence of A parameters $a_1, \ldots, a_A$; $(b_{B_k}^{(j)})$ stands for the sequence of B_k parameters $b_1^{(j)}, \ldots, b_{B_k}^{(j)}$; $i, j, k = 1, 2, \ldots, r$; with similar interpretations for (c_C) and $(d_{D_k}^{(j)})$. Colon (:) and semicolon (;) separate the forms of $(\alpha, m_1 + \ldots + m_r)$ and $(\beta_1, m_1), \ldots, (\beta_r, m_r)$. An empty product is to be treated as unity. This interpretation will be retained throughout this paper.

The hypergeometric series involved in (2.6) is a particular case of multiple hypergeometric series due to Srivastava and Daoust [1969, p. 454, (4.1)].

3. Particular Cases of (2.6)

Let $A = B_i = D_i = \delta = 1$, $C = 0$; $b_1^{(i)} = \beta_i$, $d_1^{(i)} = \gamma$, $a_1 = \lambda = \alpha$; $i = 1, 2, \ldots, r$ and using Vandermonde's theorem [Rainville 1963, p. 69, (4)

$${}_2F_1\left[\begin{matrix}-n, \beta ; \\ \gamma ;\end{matrix}\; 1\right] = \frac{(\gamma - \beta, n)}{(\gamma, n)} \qquad \ldots(3.1)$$

we get

$$(1-\Sigma Z_r)^{-\alpha} F_A\left[\begin{matrix}\alpha:\beta_1;\ldots;\beta_r;\\ -:\gamma_1;\cdots;\gamma_r;\end{matrix}\frac{-z_1}{1-\Sigma Z_r},\ldots,\frac{-z_r}{1-\Sigma Z_r}\right]$$

$$=F_A\left[\begin{matrix}\alpha:\gamma_1-\beta_1;\ldots;\gamma_r-\beta_r;\\ -:\gamma_1\quad;\ldots;\gamma_r\quad;\end{matrix}z_1,\ldots,z_r\right], \qquad \ldots(3.2)$$

where

$$|\Sigma Z_{r-1}|+|z_r|<1,\ \text{Max.}\left\{\left|\frac{z_1}{1-\Sigma Z_r}\right|+\ldots+\left|\frac{z_r}{1-\Sigma Z_r}\right|\right\}<1.$$

Formula (3.2) is evidently a generalization of the known result [Erdelyi *et al.* 1953, p. 240, (8)]

$$(1-z_1-z_2)^{-\lambda}\ F_2\left[\begin{matrix}\alpha:\beta_1;\beta_2;\\ -:\gamma_1;\gamma_2;\end{matrix}\frac{-z_1}{1-z_1-z_2},\frac{-z_2}{1-z_1-z_2}\right]$$

$$=F_2\left[\begin{matrix}\alpha:\gamma_1-\beta_1;\gamma_2-\beta_2;\\ -:\gamma_1;\quad\gamma_2;\end{matrix}z_1,z_2\right], \qquad \ldots(3.3)$$

provided $|z_1|+|z_2|<1$,

$$\text{Max.}\left\{\left|\frac{z_1}{1-z_1-z_2}\right|+\left|\frac{z_2}{1-z_1-z_2}\right|\right\}<1,$$

which itself is a generalization of well-known Euler's transformation formula [Rainville 1963, p. 60, (4)]

$$(1-z_1)^{-\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\beta_1;\\ \gamma_1\ ;\end{matrix}\frac{-z_1}{1-z_1}\right]={}_2F_1\left[\begin{matrix}\alpha,\gamma_1-\beta_1;\\ \gamma;\end{matrix}z_1\right],$$

provided

$$|z_1|<1,\left|\frac{z_1}{1-z_1}\right|<1. \qquad \ldots(3.4)$$

Taking $\delta=A=2,\ \ C=B_i=D_i-1=0;\ \ a_1=\frac{\lambda}{2},\ \ a_2=\frac{\lambda+1}{2}$,

$d_1^{(i)}=d_i;\ i=1,2,\ldots,r$ and using Vandermonde's theorem once again, to obtain

$$(1-\Sigma Z_r)^{-\lambda}\ F_C\left[\begin{matrix}\frac{\lambda}{2},\frac{\lambda+1}{2}:-;\ldots;-;\\ \quad -\quad :d_1;\ldots;d_r;\end{matrix}\frac{z_1^2}{(1-\Sigma Z_r)^2},\ldots,\frac{z_r^2}{(1-\Sigma Z_r)^2}\right]$$

$$=F_A\left[\begin{matrix}\lambda:d_1-\frac{1}{2};\ldots;d_r-\frac{1}{2};\\ -:2d_1-1;\ldots;2d_r-1;\end{matrix}2z_1,\ldots,2z_r\right] \qquad \ldots(3.5)$$

provided:

$$|\Sigma Z_{r-1}|+|z_r|<1,\left|\frac{z_1^2}{(1-\Sigma Z_r)^2}\right|^{1/2}+\ldots+\left|\frac{z_r^2}{(1-\Sigma Z_r)^2}\right|^{1/2}<1.$$

When

$$z_3 \to 0, z_4 \to 0, \ldots, z_r \to 0,$$

(3.5) reduces to the following known result due to Baily (1953)

$$(1 - z_1 - z_2)^{-\lambda} F_4 \left[\begin{matrix} \frac{\lambda}{2}, \frac{\lambda+1}{2} : - ; - ; \\ - : d_1 ; d_2 ; \end{matrix} \frac{z_1^2}{(1-z_1-z_2)^2}, \frac{z_2^2}{(1-z_1-z_2)^2}\right]$$

$$= F_2 \left[\begin{matrix} \lambda : d_1 - \frac{1}{2} ; d_2 - \frac{1}{2} ; \\ - : 2d_1 - 1 ; 2d_2 - 1 ; \end{matrix} 2z_1, 2z_2\right], \qquad \ldots(3.6)$$

provided :

$$|z_1| + |z_2| < 1, \left|\frac{z_1^2}{(1-z_1-z_2)^2}\right|^{1/2} + \left|\frac{z_2^2}{(1-z_1-z_2)^2}\right|^{1/2} < 1,$$

which is in fact a generalization of the known quadratic transformation formula [Rainville 1963, p. 65, (1)]:

$$(1 - z_1)^{-\lambda} {}_2F_1 \left[\begin{matrix} \frac{\lambda}{2}, \frac{\lambda+1}{2} \\ d_1 + \frac{1}{2} \end{matrix} \frac{z_1^2}{(1-z_1)^2}\right] = {}_2F_1 \left[\begin{matrix} \lambda, d_1 \\ 2d_1 \end{matrix} 2z_1\right], \qquad \ldots(3.7)$$

provided

$$|z_1| < 1, \left|\frac{z_1^2}{(1-z_1)^2}\right|^{1/2} < 1.$$

Finally, let $r = 2$ so that we obtain

$$(1 - z_1 - z_2)^{-\lambda} F \left[\begin{matrix} (a_A) \cdot (b_B) ; (b'_{B'}) ; \\ (c_C) : (d_D) ; (d'_{D'}) ; \end{matrix} \left(\frac{-z_1}{1-z_1-z_2}\right)^{\delta}, \left(\frac{z_2^2}{1-z_1-z_2}\right)^{\delta}\right]$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(\lambda, n_1 + n_2)}{n_1 ! n_2 !} z_1^{n_1} z_2^{n_2}$$

$$\times F \left[\begin{matrix} (a_A) : (b_B), \Delta(\delta, -n_1) ; (b'_{B'}), \Delta(\delta, -n_2) ; \\ (c_C), \Delta(\delta, \lambda) : (d_D) ; (d'_{D'}) ; \end{matrix} 1, 1\right],$$

$$\ldots(3.8)$$

which is the main formula in (Singal 1972, p. 201, 2.1).

ACKNOWLEDGEMENT

The author wishes to express his sincere thanks to Dr. R. C. Verma for his kind supervision during the preparation of this paper.

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