COLLINEATION AND MOTION IN AN ELECTROMAGNETIC FIELD WITH INDUCTION

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In this paper it has been shown that Ricci collineation along flow vector implies motion in an electromagnetic field with induction. It has also been shown that the Killing flow vector and the Killing magnetic field vector do not propagate with the fundamental velocity.

We will first study the system of equations in an electromagnetic field with induction. Inside matter electromagnetic field is defined by two skew-symmetric tensor fields H and G of order 2.

H is called the electric field magnetic induction tensor and the tensor G the electric induction magnetic field tensor. Let $*H$ and $*G$ be the dual 2 forms of H and G in the space time V_4. If η is the volume element 4-form of V_4 corresponding to this orientation, we have

$$*H_{\alpha\beta} = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta}\, H^{\gamma\delta},$$

$$*G_{\alpha\beta} = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta}\, G^{\gamma\delta}.$$

It has been shown by Lichnerowicz (1967) that

$$*H_{\alpha\beta} = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta} H^{\gamma\delta} = u_\alpha b_\beta - u_\beta b_\alpha + \eta_{\alpha\beta\lambda\mu} u^\lambda e^\mu,$$

$$*G_{\alpha\beta} = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta} G^{\gamma\delta} = u_\alpha h_\beta - u_\beta h_\alpha + \eta_{\alpha\beta\lambda\mu} u^\lambda d^\mu,$$

where u is a unitary vector defining a time-like direction and

h_α = magnetic field or intensity vector,

b_α = magnetic induction vector,

e_α = electric field or intensity vector,

d_α = electric induction vector.

These four vectors are orthogonal to u, i.e.

$$u^\alpha e_\alpha = u^\alpha d_\alpha = u^\alpha d_\alpha = u^\alpha h_\alpha = 0.$$

The energy momentum tensor is calculated by eliminating time derivatives of electric and magnetic fields from Maxwell field equations. It is given by

$$T_{\alpha\beta} = \mu\, [(\tfrac{1}{2}\, g_{\alpha\beta} - u_\alpha u_\beta)\, h_\rho h^\rho - h_\alpha h_\beta]. \qquad \ldots(1)$$

Here $b = \mu h$

μ being magnetic permeability and h_p the space-like vector.

$$T_{ab}g^{ab} = T = \mu h_\rho h^\rho. \qquad ...(2)$$

But $$R_{ab} = T_{ab} - \tfrac{1}{2} T g_{ab}. \qquad ...(3)$$

Substituting the values of T_{ab} and T in (3) we obtain

$$R_{ab} = -\mu\,[u_a u_b + h_a h_b]. \qquad ...(4)$$

Theorem 1—In electromagnetic field, if magnetic field vector is harmonic and permeability is uniform and the flow vector is Born rigid, then

$$\underset{u}{\mathcal{L}} R_{ab} = 0 \Leftrightarrow \underset{u}{\mathcal{L}} g_{ab} = 0. \qquad ...(5)$$

PROOF :

$$\underset{u}{\mathcal{L}} R_{ab} = R_{ab;k}u^k + R_{kb}u^k_{;a} + R_{ak}u^k_{;b}$$

$$= \mu\,[u_a u_b + h_a h_b]_{;k}u^k + \mu\,[u_k u_b + h_k h_b]\,u^k_{;a}$$
$$+ \mu\,[u_a u_k + h_a h_k]\,u^k_{;b} \qquad ...(6)$$

$$= \mu u_{a;k}u^k u_b + \mu u_a u_{b;k}u^k + \mu h_b h_{a;k}u^k$$
$$+ \mu h_{b;k}h_a u^k + \mu u_k u_b u^k_{;a} + \mu h_k h_b u^k_{;a} + \mu u_a u_k u^k_{;b}$$
$$+ \mu h_a h_k u^k_{;b} + \mu_{;k}u_a u_b u^k + \mu_{;k}h_a h_b u^k \qquad ...(7)$$

$$= \mu_{;k}u_a u_b u^k + \mu_{;k}h_a h_b u^k + \mu h_b u^k (h_{a;k} - h_{k;a})$$
$$+ h_a u^k \mu (h_{b;k} - h_{k;b}) + \mu \dot{u}_a u_b + \mu \dot{u}_b u_a. \qquad ...(8)$$

We know that h_p is harmonic. Therefore $h_{a;b} - h_{b;a} = 0$. Substituting from this equation in (8) we obtain

$$\underset{u}{\mathcal{L}} R_{ab} = \mu_{;k}u_a u_b u^k + \mu_{;k}h_a h_b u^k + \mu \dot{u}_a u_b + \mu \dot{u}_b u_a \qquad ...(9)$$

Since permeability is uniform, $\mu_{;k}u^k = 0$. Consequently (9) becomes

$$\underset{u}{\mathcal{L}} R_{ab} = \mu \dot{u}_a u_b + \mu \dot{u}_b u_a. \qquad ..(10)$$

If the flow is Born rigid then

$$\underset{u}{\mathcal{L}} h_{ab} = 0,$$

that is, $\underset{u}{\mathcal{L}} (g_{ab} - u_a u_b) = 0$, or $\quad u_{a;b} + u_{b;a} = \dot{u}_a u_b + \dot{u}_b u_a.$ ...(11)

Hence $\underset{u}{\mathcal{L}} R_{ab} = 0 \Rightarrow \mu (\dot{u}_a u_b + \dot{u}_b u_a) = 0,$ by (11), and $\mu \neq 0$

$$\mu (u_{a;b} + u_{b;a}) = 0.$$

i.e.,

Therefore $\underset{u}{\mathcal{L}} g_{ab} = 0$

that is Ricci collineation implies motion.

Remark : In an electromagnetic field, magnetic field vector is harmonic and permeability is uniform. Ricci collineation implies motion along the flow vector, where flow is assumed to be Born rigid.

Theorem 2—In electromagnetic field with induction the Killing flow vector does not propagate with the fundamental velocity.

PROOF : If u^a propagates with the fundamental velocity then $\Box u_a = 0$. Besides u^a is a Killing vector. We know that

$$u^a R_{ab} = 0$$

that is $\quad - \mu [u_a u_b + h_a h_b] u^a = 0$

or $\quad - \mu u_b = 0 \Leftrightarrow$

$$- \mu = 0 \quad \text{as} \quad u_b \neq 0$$

which is physically incompatible since μ, the magnetic permeability, is positive.

Theorem 3—In electromagnetic field with induction, the Killing magnetic field vector does not propagate with the fundamental velocity.

PROOF : If h^a propagates with the fundamental velocity $\Box h^a = 0$.

Besides, if h^a is a Killing vector then $u^a R_{ab} = 0$, that is

$$- \mu [u_a u_b + h_a h_b] h^a = 0$$

$$- \mu h_b = 0, \quad - \mu = 0 \quad \text{as} \quad h_b \neq 0,$$

which is physically incompatible since μ is positive.

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