MEAN MOTIONS AND CHARACTERISTIC EXPONENTS AT COLLINEAR LIBERATION POINTS WHEN THE BIGGER PRIMARY IS AN OBLATE SPHEROID

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The oblateness of the bigger primary contributes an additional term in the poten tial, involving the moment of inertia of the primary. As a result, the roots of the characteristic equation at the liberation points of the restricted problem depend not only upon the mass parameter μ, but also on the oblateness of the spheroid.

1. Introduction

In this paper the effect of oblateness of the bigger primary on the location of collinear liberation points and mean motions and characteristic exponents at these points, is considered. This effect appears as an additional term in the potential due to the bigger primary and this leads to a seventh order algebraic equation in ξ, the distance of liberation point from one of the primaries. The root ξ is worked out as a power series in μ and certain other terms arising out of oblateness and depending upon $I=\frac{1}{2}(I_3-I_e) > 0$, where I_3 is the moment of inertia of the oblate spheroid about an axis normal to the equatorial plane, and I_e is its moment of inertia about any axis in the equatorial plane. The term I, when expressed in dimensionless synodicco-ordinates, is a very small quantity; of the order of $3{\cdot}64 \times 10^{-7}$ for Earth-Moon system and 7×10^{-18} for Sun-Earth system.

Therefore, we have neglected second and higher order terms in I.

The characteristic roots are then determined from the fourth-order characteristic equation using the analytic expressions for ξ_i ($i=1,2,3$) and so the frequencies of the oscillatory and exponential mode for the system are determined.

This paper, in fact, is a generalization of the results of Szebehely (1970) and those results can be derived merely by putting $I=0$.

2. Equations of Motion

In dimensionless synodic co-ordinates, the equations of motion of the infinitesimal mass under the gravitational influence of the two primaries may be written as

$$\ddot{x}-2\dot{y}=\frac{\partial \Omega}{\partial x}$$

$$\ddot{y}+2\dot{x}=\frac{\partial \Omega}{\partial y} \qquad \ldots \quad \ldots \quad \ldots \qquad (1)$$

where the bigger primary is an oblate spheroid so that

$$\Omega = \tfrac{1}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{I_3-I_e}{2r_1^3} \qquad \ldots \qquad (2)$$

with $\quad r_1^2 = (x-\mu)^2+y^2 \quad ; \quad r_2^2 = (x+1-\mu)^2 + y^2,$

I_3, the moment of inertia of the oblate spheroid about z-axis, and I_e, its moment of intertia about any axis in the equatorial plane (McCusky 1963).

Writing $I_3-I_e= 2I > 0$,

we can write (2) as

$$\Omega = \tfrac{1}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{I}{r_1^3} \qquad \ldots \quad \ldots \qquad (3)$$

3. Location of Liberation Points

At the liberation points, we have

$$\frac{\partial\Omega}{\partial x} = 0 \quad ; \quad \frac{\partial\Omega}{\partial y} = 0$$

or, $$x-\frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} - \frac{3I(x-\mu)}{r_1^5} = 0 \qquad \ldots \qquad (4)$$

and $$y\left[1-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3I}{r_1^5}\right] = 0 \qquad \ldots \quad \ldots \quad \ldots \qquad (5)$$

At collinear liberation points, $y=0$ and the abscissa is given by eqn. (4). Denoting the expression of eqn. (4) by $f(x)$, we may observe that

$f(-\infty) <0 \; ; f(\mu-1) >0 \; ; f(\mu) <0 \; ; f(\infty)> 0$;

so that the three collinear liberation points L_1, L_2, L_3 lie in the intervals $-\infty$ and $(\mu-1)$; $(\mu-1)$ and μ; μ and $+\infty$ respectively.

(a) *The point* L_1

Since the point L_1 lies between $-\infty$ and $(\mu-1)$, i.e. on the left of the smaller primary, we have, for this point

$$r_1 = \mu - x \; ; \; r_2 = \mu - 1 - x,$$

which, when substituted in (4), gives

$$x + \frac{1-\mu}{(x-\mu)^2} + \frac{\mu}{(x+1-\mu)^2} + \frac{3I}{(x-\mu)^4} = 0 \qquad \ldots \quad \ldots \quad \ldots \qquad (6)$$

This is a seventh order algebraic equation in x.

Putting $r_2 = \mu - 1 - x = \xi_1$, eqn. (6) becomes

$$(\mu-1-\xi_1) + \frac{1-\mu}{(1+\xi_1)^2} + \frac{\mu}{\xi_1^2} + \frac{3I}{(1+\xi_1)^4} = 0 \qquad \ldots \quad \ldots \qquad (7)$$

Or, $\xi_1^7 + (5-\mu)\xi_1^6 + (10-4\mu)\xi_1^5 + (9-6\mu)\xi_1^4 + (3-6\mu)\xi_1^3 - (6\mu+3I)\xi_1^2 - 4\mu\xi_1 - \mu = 0$

$$\ldots \qquad (8)$$

Equation (8) is a seventh order algebraic equation in ξ_1, for the point L_1. Since there is only one change of sign, according to Descarte's sign rule, there will be at least one real root. And as the left hand side of equation (8) is <0 for $\xi_1=0$ and $+\infty$ for $\xi_1=\infty$, this real root must be a positive root. We may note that there is one real root ($\xi_1=0$) for $\mu=0$.

For solving eqn. (7) in the power series of μ, we may take starting value as $\xi_1=[\mu/3(1-\mu)]^{1/3}$ (Szebehely 1967).

We may write equation (7) as

$$\frac{\mu}{3(1-\mu)} = \frac{\xi_1^3\,[\,1+\xi_1+\frac{1}{3}\,\xi_1^2 - \lambda'\,(1+\xi_1)^{-2}]}{(1+\xi_1)^2\,[1-\xi_1^3+3I\xi_1^2\,(1+\xi_1)^{-4}]} \qquad \ldots \quad \ldots \quad \ldots \qquad (9)$$

where $\lambda' = I/\xi_1$, is an infinitesimal, as I is very small in comparison with ξ_1.

Writing

$$\nu^3 = \frac{\mu}{3(1-\mu)} \qquad \ldots \quad \ldots \quad \ldots \qquad (10)$$

we may express ν as a power series in ξ_1, from eqn. (9), as

$$\nu = \xi_1 \left[1 - \frac{1}{3}\xi_1 + \frac{1}{3}\,\xi_1^2 + \frac{1}{81}\,\xi_1^3 + \frac{47}{243}\,\xi_1^4 - \frac{129}{729}\,\xi_1^5 \right]$$

$$-\lambda'\xi_1\left[\frac{1}{3}-\frac{10}{9}\,\xi_1+\frac{7}{3}\xi_1^2-\frac{692}{243}\,\xi_1^3+\frac{912}{729}\,\xi_1^4+\frac{17253}{2187}\,\xi_1^5\right]$$

+ higher order terms.

$$\text{or,}\quad \nu=\xi_1\left[1-\frac{1}{3}\,\xi_1+\frac{1}{3}\,\xi_1^2+\frac{1}{81}\xi_1^3+\frac{47}{243}\xi_1^4-\frac{129}{729}\,\xi_1^5\right]$$

$$-I\left[\frac{1}{3}-\frac{10}{9}\,\xi_1+\frac{7}{3}\,\xi_1^2-\frac{692}{243}\xi_1^3+\frac{912}{729}\xi_1^4+\frac{17253}{2187}\,\xi_1^5\right]$$

+ higher order terms.

$$\Rightarrow\quad \xi_1=\left(\nu+\frac{1}{3}I\right)-\left(-\frac{1}{3}\xi_1^2+\frac{10}{9}I\xi_1\right)-\left[\frac{1}{3}\xi_1^3-\frac{7}{3}I\xi_1^2\right]-\left[\frac{1}{81}\xi_1^4+\frac{692}{243}I\xi_1^3\right]$$

$$-\left[\frac{47}{243}\xi_1^5-\frac{912}{729}I\xi_1^4\right]+\left[\frac{129}{729}\xi_1^6+\frac{17253}{2187}I\xi_1^5\right]$$

+ higher order terms.

As a first approximation, we have

$$\xi_1=\nu+\tfrac{1}{3}I.$$

Following Largrange's method of inversion of series, we obtain

$$\xi_1=\nu\left[1+\frac{1}{3}\nu-\frac{1}{9}\nu^2-\frac{31}{81}\nu^3-\frac{119}{243}\nu^4-\frac{1}{9}\nu^5\right]$$

$$+\tfrac{1}{3}I\left[1-\frac{8}{3}\nu+\frac{10}{3}\nu^2-\frac{481}{81}\nu^4-\frac{5155}{243}\nu^5\right.$$

+ higher order terms. .. .. (11)

where ν, as given by eqn. (10) is

$$\nu=[\mu/3(1-\mu)]^{\frac{1}{3}}\simeq[\mu/3]^{\frac{1}{3}}.$$

It is clear that result of Szebehely (1970) can be deduced from equation (11), merely by putting $I=0$.

(b) *The point* L_2

Since the point L_2 lies between $\mu-1$ and μ, i.e. between the two primaries, we have, for this point

$$r_1=\mu-x \qquad \text{and} \qquad r_2=1-\mu+x$$

which, when substituted in eqn. (4) give

$$x+\frac{1-\mu}{(x-\mu)^2}-\frac{\mu}{(x+1-\mu)^2}+\frac{3I}{(x-\mu)^4}=0. \qquad \ldots \qquad \ldots \qquad (12)$$

Writing $r_2 = 1-\mu+x = \xi_2$, we can write eqn. (12) as

$$(\xi_2+\mu-1)+\frac{1-\mu}{(-1+\xi_2)^2}-\frac{\mu}{\xi_2^2}+\frac{3I}{(1-\xi_2)^4}=0 \qquad \ldots \qquad \ldots \qquad (13)$$

or, $$\xi_2^7-(5-\mu)\xi_2^6+2(5-2\mu)\xi_2^5-(9-4\mu)\xi_2^4+(3+2\mu)\xi_2^3-3(2\mu-I)\xi_2^2+4\mu\xi_2-\mu=0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (14)$$

Descarte's sign rule, when applied to (14), shows the existence of at least one positive root. Proceeding as before, we get

$$\nu=\xi_2\left[1+\frac{1}{3}\xi_2+\frac{1}{3}\xi_2^2+\frac{53}{81}\xi_2^3+\frac{101}{243}\xi_2^4+\frac{291}{729}\xi_2^5\right]$$

$$+\tfrac{1}{3}\lambda''\xi_2\left[1+\frac{10}{3}\xi_2+7\xi_2^2+\frac{1232}{81}\xi_2^3+\frac{7760}{243}\xi_2^4+\frac{15080}{243}\xi_2^5\right]$$

$+$ higher order terms. $\qquad \lambda''=I/\xi_2$

Or, $$\nu=\xi_2\left[1+\frac{1}{3}\xi_2+\frac{1}{3}\xi_2^2+\frac{53}{81}\xi_2^3+\frac{101}{243}\xi_2^4+\frac{291}{729}\xi_2^5\right]$$

$$+\tfrac{1}{3}I\left[1+\frac{10}{3}\xi_2+7\xi_2^2+\frac{1232}{81}\xi_2^3+\frac{7760}{243}\xi_2^4+\frac{15080}{243}\xi_2^5\right]$$

$+$ higher order terms.

$$\Rightarrow \quad \xi_2=\left[\nu-\frac{1}{3}I\right]-\left[\frac{1}{3}\xi_2^2+\frac{10}{9}I\xi_2\right]-\left[\frac{1}{3}\xi_2^3+\frac{7}{3}I\xi_2^2\right]-\left[\frac{53}{81}\xi_2^4+\frac{1232}{243}I\xi_2^3\right]$$

$$-\left[\frac{101}{243}\xi_2^5+\frac{7760}{729}I\xi_2^4\right]-\left[\frac{291}{729}\xi_2^6+\frac{15080}{729}I\xi_2^5\right]$$

$+$ higher order terms.

Again, as a first order approximation,

$$\xi_2=\nu-\tfrac{1}{3}I$$

and following Lagrange's method of inversion of series, we obtain

$$\xi_2=\nu\left[1-\frac{1}{3}\nu-\frac{1}{9}\nu^2-\frac{23}{81}\nu^3+\frac{151}{243}\nu^4-\frac{1}{9}\nu^5\right]$$

$$-\tfrac{1}{3}I\left[1+\frac{8}{3}\nu+\frac{10}{3}\nu^2+4\nu^3+\frac{1355}{243}\nu^4+\frac{818}{243}\nu^5\right]$$

$+$ higher order terms. .. .. (15)

where ν is given by eqn. (10).

It also follows from this equation (15) that the second series is due to oblateness of the primary and the results of Szebehely (1970) can be deduced by putting $I=0$.

(c) *The Point L_3*

Since the third collinear liberation point L_3 lies between μ and $+\infty$, i.e. to the right of the bigger primary, we have, for this point

$$r_1=x-\mu \quad \text{and} \quad r_2=x+1-\mu\ .$$

Substituting in (4), we get

$$x-\frac{1-\mu}{(x-\mu)^2}-\frac{\mu}{(x+1-\mu)^2}-\frac{3I}{(x-\mu)^4}=0 \quad .. \quad (16)$$

Writing $r_1=x-\mu=\xi_3$, equation (16) becomes

$$(\mu+\xi_3)-\frac{1-\mu}{\xi_3^2}-\frac{\mu}{(1+\xi_3)^2}-\frac{3I}{\xi_3^4}=0 \quad .. \quad .. \quad (17)$$

or, $\xi_3^7\,(2+\mu)\,\xi_3^6+(1+2\mu)\,\xi_3^5-(1-\mu)\,\xi_3^4-2\,(1-\mu)\,\xi_3^3-(1-\mu+3I)\,\xi_3^2$
$$-6I\xi_3-3I=0 \quad .. \quad (18)$$

Equation (18) is the seventh order algebraic equation in ξ_3 for the point L_3. Since there is only one change of sign, Descarte's sign rule shows the existence of at least one real root. Since the left-hand side of eqn. (18) is < 0 for $\xi_3=0$ and $+\infty$ for $\xi_3=\infty$, therefore, this real root must be a positive root. Moreover, since this root (ξ_3) is near to $+1$, it is advantageous to introduce a new variable η defined as

$$\eta=\xi_3-1$$

subject to which eqn. (18) becomes

$$\eta^7+(9+\mu)\,\eta^6+2\,(17+4\mu)\,\eta^5+(69+26\mu)\,\eta^4+(79+46\mu)\,\eta^3+3\,(16+16\mu-I)\,\eta^2+4\,(3+7\mu-3I)\,\eta+7\mu-12I=0. \quad .. \quad .. \quad (19)$$

The first approximation for η follows from linear part of eqn. (19), viz.

$$12\,\eta+7\mu-12I=0$$

$$\Rightarrow \qquad \eta=-\frac{7}{12}\,\mu+I$$

$$=-\nu_1+I \quad .. \quad .. \quad .. \quad (20)$$

where $\nu_1=7\mu/12$. .. .. .. (21)

The existence of a negative root is indicated by the seven sign changes occuring in (19) when a change $\eta\to-\eta$ is made.

Equation (20) serves as a starting value for η Following the method of successive approximations, we get

$$\eta=-\nu_1\left[1+\frac{23}{84}\,\nu_1^2+\frac{23}{84}\,\nu_1^3\right]+I\left[1+3\,\nu_1+\frac{87}{14}\,\nu_1^2-\frac{209}{21}\,\nu_1^3\right]$$

$+$ higher order terms.

$$\therefore \ \xi_3 = 1 - \nu_1 \left[1 + \frac{23}{84} \nu_1^2 + \frac{23}{84} \nu_1^3 \right] + I \left[1 + 3\nu_1 + \frac{87}{14} \nu_1^2 - \frac{209}{21} \nu_1^3 \right]$$

+ higher order terms,

or, using eqn. (21),

$$\xi_3 = 1 - \frac{7}{12} \mu \left[1 + \frac{1127}{12096} \mu^2 + \frac{7889}{145152} \mu^3 \right]$$

$$+ I \left[1 + \frac{7}{4} \mu + \frac{4263}{2016} \mu^2 - \frac{71687}{36288} \mu^3 \right] + \text{higher order terms.} \qquad (22)$$

Here again, it is noticed that results of Szebehely (1970) can be deduced by putting $I=0$.

Equations (11), (15) and (22) give the location of the collinear equilibrium points L_1, L_2, L_3 respectively in terms of a power series in μ and terms involving I.

4. The Characteristic Equation

The characteristic roots are obtained from

$$\lambda^4 + (4 - \Omega^\circ_{xx} - \Omega^\circ_{yy})\lambda^2 + \Omega^\circ_{xx}\,\Omega^\circ_{yy} - (\Omega^\circ_{xy})^2 = 0 \text{ (Szebehely 1967).} \qquad (23)$$

From (3), we have

$$\Omega = \frac{1}{2} \left[(1-\mu)\, r_1^2 + \mu r_2^2 \right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{I}{r_1^3}$$

Double differentiation with respect to x and y yields

$$\Omega^\circ_{xx} = 1 + \frac{2(x-\mu)^2 (1-\mu)}{r_1^5} + \frac{2\mu\,(x+1-\mu)^2}{r_2^5} + \frac{12I\,(x-\mu)^2}{r_1^7},$$

$$\Omega^\circ_{yy} = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3I}{r_1^5},$$

$$\Omega^\circ_{xy} = \textit{Zero} = \Omega^\circ_{yx}.$$

Now, as defined by Szebehely (1970), we take

$$\phi_1 = \frac{1-\mu}{(1+\xi_1)^3} + \frac{\mu}{\xi_1^3}$$

$$\phi_2 = \frac{1-\mu}{(-\xi_2+1)^3} + \frac{\mu}{\xi_2^3} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

$$\phi_3 = \frac{\mu}{(1+\xi_3)^3} + \frac{1-\mu}{\xi_3^3}$$

and introduce ψ_i, $i = 1, 2, 3$, as

$$\psi_1 = \frac{3I}{(1+\xi_1)^5} \;,\; \psi_2 = \frac{3I}{(1-\xi_2)^5} \;;\; \psi_3 = \frac{3I}{\xi_3^5} \quad \ldots \quad \ldots \quad (25)$$

Then

$$\Omega^0_{xx}\,(L_i) = 1+2\phi_i+4\psi_i,$$

$$\Omega^0_{yy}\,(L_i) = 1-\phi_i-\psi_i\,, \qquad (i = 1, 2, 3)$$

$$\Omega^0_{xy}\,(L_i) = \text{Zero}$$

The characteristic eqn. (23) for the point L_1 becomes

$$\lambda^4+(2-\phi_1-3\psi_1)\lambda^2 + (1+2\phi_1+4\psi_1)(1-\phi_1-\psi_1) = 0$$

Writing $\lambda^2 = \gamma$, we get a quadratic in γ and its two roots are given as

$$\gamma_{1,2} = -1+\tfrac{1}{2}(\phi_1+3\psi_1) \pm \tfrac{1}{2}\{(3\phi_1+5\psi_1)^2 - 8(\phi_1+3\psi_1)\}^{\frac{1}{2}}$$

$\therefore$ S_1, the real frequency of oscillatory mode

$$= [-\gamma_2]^{\frac{1}{2}}$$

$$= [1-\tfrac{1}{2}(\phi_1+3\psi_1)+\tfrac{1}{2}\{(3\phi_1+5\psi_1)^2 - 8(\phi_1+3\psi_1)\}^{\frac{1}{2}}]^{\frac{1}{2}}$$

and λ_1, that of exponential mode,

$$= [\gamma_1]^{\frac{1}{2}}$$

$$= [-1+\tfrac{1}{2}(\phi_1+3\psi_1)+\tfrac{1}{2}\{(3\phi_1+5\psi_1)^2 - 8(\phi_1+3\psi_1)\}^{\frac{1}{2}}]^{\frac{1}{2}}\,.$$

Similarly for the points L_2 and L_3, we have

$$S_2 = [1-\tfrac{1}{2}(\phi_2+3\psi_2)+\tfrac{1}{2}\{(3\phi_2+5\psi_2)^2 - 8(\phi_2+3\psi_2)\}^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$\lambda_2 = [-1+\tfrac{1}{2}(\phi_2+3\psi_2)+\tfrac{1}{2}\{(3\phi_2+5\psi_2)^2 - 8(\phi_2+3\psi_2)\}^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$S_3 = [1 - \tfrac{1}{2}(\phi_3+3\psi_3)+\tfrac{1}{2}\{(3\phi_3+5\psi_3)^2 - 8(\phi_3+3\psi_3)\}^{\frac{1}{2}}]^{\frac{1}{2}}$$

and $\lambda_3 = [-1+\tfrac{1}{2}(\phi_3+3\psi_3)+\tfrac{1}{2}\{(3\phi_3+5\psi_3)^2 - 8(\phi_3+3\psi_3)\}^{\frac{1}{2}}]$.

Thus, for L $(i = 1, 2, 3)$, we see that

$$S_i = [1 - \tfrac{1}{2}(\phi_i+3\psi_i)+\tfrac{1}{2}\{(3\phi_i+5\psi_i)^2 - 8(\phi_i+3\psi_i)\}^{\frac{1}{2}}]^{\frac{1}{2}}$$

and $\lambda_i = [-1+\tfrac{1}{2}(\phi +3\psi_i)+\tfrac{1}{2}\{(3\phi_i+5\phi_i)^2 - 8(\phi_i+3\psi_i)\}^{\frac{1}{2}}]^{\frac{1}{2}}$

where ϕ_i's and ψ_i's are given by eqns. (24) and (25).

The values of ξ_i as given by eqns. (11), (15) and (22) are substituted in the equations for respective ϕ_i and ψ_i and these values of ϕ_i's and ψ_i's are put in equations (26) and (27) to get the mean motions and characteristic exponents at collinear liberation points. It is clear that $I = 0$ would imply $\psi_i = 0$ and the results of Szebehely (1970) would at once follow.

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References

Berezin, I. S., and Zhidkove, N. P. (1965). Computing Methods, Vol. II. Addison-Wesley Publishing Company Inc., London, pp. 79-84.

McCusky, S. W. (1963). Introduction to Celestial Mechanics, Addison-Wesley Publishing Company, Inc., London, pp. 163-67.

Szebehely, V. (1967). Theory of Orbits. Academic Press, Inc., New York.

Szebehely, V., Ingram, D. S., and Cochran, J. E. (1970). Mean motions and characteristic exponents at the liberation points. *Astr. J.*, **75**, 92.