SIMILARITY SOLUTIONS FOR EXPLOSIONS IN RADIATING STARS

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Similarity solutions for central explosions in stars, with radiation heat flux, have been obtained under the assumption of isothermal shock conditions. The density of the gaseous configuration is taken to obey a power law. Such types of solutions exist only when radiation pressure and energy are both neglected. Numerical solutions for the case of constant total energy of the wave are tabulated.

1. Introduction

The gas dynamical problem of explosions in stars has been treated by several authors. Many of them particularly, Carrus *et al.* (1951), Sedov (1959), Deb Ray (1965) and Bhatnagar *et al.* (1965), have studied the similarity behaviour of the flow parameters for central stellar explosions, without taking the effect of radiation. Bhatnagar and Kushwaha (1961), following the method of Brinkley and Kirkwood (1947), however, have investigated the non-self-similar problem with radiation pressure and energy, while ignoring the flux of radiation. They have taken the external shock surface as transparent and isothermal. The validity of their solutions is confined to the neighbourhood of the external shock boundary.

Similarity solutions for a nuclear explosion in air, under different physical assumptions, have been studied in details by Elliot (1960).

In this paper, we give the results of our study for the existence of self-similar solutions with radiation for stellar explosions. It is found that under certain postulates, such solutions can exist only in those corresponding models without radiation and that too, when we take the effect of radiation flux alone. We assume for our study the shock as transparent and isothermal, while the gas as gray and opaque. Initially, the gas is at rest and its radiation ineffective compared to those in the after-explosion stage. Appropriate range of temperature of stars for such solutions is feasible. This implies that the temperature is high enough for consideration of the radiation heat flux but sufficiently low for the radiation pressure and energy to be effective. This is all the more justified as a considerable part of the inner region with very high temperature and acting like an expanding piston, falls outside the zone of our interest.

Numerical results in a few cases for the model, corresponding to instantaneous central explosion have been tabulated. Here, as explained by Hazlehurst (1962), for "instantaneous" we should read "in a time much less than that taken by a sound wave to reach the surface" and for "central" we should substitute "considerably more than a fifth of a stellar radius below the surface."

2. Equations of Motion and Boundary Conditions

The equations of flow behind a spherical shock are

$$\frac{d\rho}{dt} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(ur^2) = 0 \qquad \text{(1)}$$

$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{G\,m}{r^2} = 0 \qquad \text{(2)}$$

$$\frac{dE}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right) + \frac{1}{\rho r^2}\frac{\partial}{\partial r}(Fr^2) = 0 \qquad \text{(3)}$$

$$\frac{\partial m}{\partial r} = 4\,\pi\rho r^2 \qquad \text{(4)}$$

where

$$\left.\begin{aligned} E &= E_m + E_R \\ p &= p_m + p_R \end{aligned}\right\} \qquad \text{(5)}$$

and

E_m being the internal energy per unit mass, and E_R the radiation energy per unit mass; p is the total pressure with p_m the material pressure and p_R the radiation pressure; F is the heat flux; u the material velocity, ρ the density, and m the mass within a radius r at time t.

For an ideal gas we have

$$\left.\begin{aligned} E_m &= \frac{p_m}{\rho\,(\gamma-1)}, \\ p_m &= \Gamma\rho\, T, \end{aligned}\right\} \qquad \text{(6)}$$

where T is the temperature, Γ the gas constant and γ the ratio of specific heats.

Also, assuming local thermodynamic equilibrium and taking Rosseland's diffusion approximation, we have

$$\rho\, E_R = 3p_R = a_1\, T^4 \qquad \text{(7)}$$

and

$$F = \frac{-c\mu}{3} \frac{\partial}{\partial r} (a_1 T^4) \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where $a_1c/4$ is the Steffan-Boltzman constant, c is the velocity of light and μ the mean-free path of radiation, being a function of density and temperature.

After Wang (1966), we take

$$\mu = \mu_0 \rho^{\alpha'} T^{\beta'} \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

μ_0, α' and β' being constants.

The disturbance is headed by an isothermal shock and the conditions are

$$\rho_1 V = \rho_2 (V - u_2) = m_s \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

$$p_2 - p_1 = m_s u_2 \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$E_1 + \frac{p_1}{\rho_1} + \frac{1}{2} V^2 = E_2 + \frac{p_2}{\rho_2} + \frac{1}{2} (V - u_2)^2 - \frac{F_2}{m_s} \quad \ldots \quad \ldots \quad (12)$$

$$T_1 = T_2 \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$m_1 = m_2 \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

the suffix 1 and 2 are for the regions just outside and just inside the shock surface respectively and V denotes the shock velocity.

Equation (13) shows that the temperature on both sides are equalised as the flow of radiation across the transparent shock surface is very rapid and so a part of the radiation energy escapes into the interstellar space almost instantaneously. The flux of radiation, however, is given by eqn. (8). This represents the continuity of the flow of radiation in the disturbed region of the stellar envelope, valid up to the shock surface.

In front of the shock, in the undisturbed gaseous medium, we have, by our assumption

$$\rho_1 = \beta R^{\alpha} \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where R is the shock radius at time t.

Since the fluid originally is in hydrostatic equilibrium, we get,

$$p_1 = \frac{-2\pi\beta^2 G R^{2+2\alpha}}{(\alpha+1)(3+\alpha)}, \; -3 < \alpha < -1. \quad \ldots \quad \ldots \quad (16)$$

Let us seek solutions of the equations in the form

$$\left.\begin{aligned} u &= \frac{r}{t}\, U(\eta) \\ \rho &= r^{K}\, t^{\lambda}\, \Omega(\eta) \\ p &= r^{K+2}\, t^{\lambda-2}\, P(\eta) \\ m &= r^{K+3}\, t^{\lambda}\, Z(\eta) \\ F &= r^{K+3}\, t^{\lambda-3}\, \Upsilon(\eta) \\ \text{where} & \\ \eta &= r^{a}\, t^{b} \end{aligned}\right\} \qquad (17)$$

Here k, λ, a and b are constants.

3. Solutions of the Equations

We choose the shock surface to be given by η_0 = constant.

Similarity conditions are compatible only when we neglect radiation pressure and radiation energy. As in ordinary gas dynamics, by direct substitutions of the form (17) in the equations of motion and shock conditions we find that we should take $k = 0$, $\lambda = -2$ besides, $\alpha' = \frac{1}{2}$ and $\beta' = -2$.

Henceforth, the total pressure, p, and the energy per unit mass, E, will stand for the material pressure and the material energy per unit mass respectively.

The mean free paths of radiation as given by eqn. (9) for these appropriate values of α' and β' are physically tenable. The wide range of values provided for Rosseland's mean free path valid for an optically thick gray gas (Penner and Olfe 1968 and Koch 1965) suggests that quite few of such photon mean free paths are included in our formula.

The equations of motion (1)—(4) and (8) when transformed may be written as follows :

$$\frac{\Omega'(\eta)}{\Omega(\eta)} = \frac{2-3\,U(\eta)}{\eta\,[b+a\,U(\eta)]} - \frac{a\,U'(\eta)}{b+a\,U(\eta)} \qquad (19)$$

$$\frac{P'(\eta)}{P(\eta)} = \frac{-2}{a\eta} + \frac{\Omega(\eta)}{a\eta\,P(\eta)}\Big[\,U(\eta)\Big\{1-U(\eta)\Big\} - \eta U'(\eta)\Big\{b+a\,U(\eta)\Big\} - G\,Z(\eta)\Big] \qquad (20)$$

$$\frac{\Upsilon'(\eta)}{\Upsilon(\eta)} = -\frac{5}{a\eta} + \frac{P(\eta)}{a\eta\,\Upsilon(\eta)}$$

$$\left\{\frac{2(1-U(\eta))-(b+aU(\eta))\eta\left(\dfrac{P'(\eta)}{P(\eta)}-\dfrac{\gamma\,\Omega'(\eta)}{\Omega(\eta)}\right)}{\gamma-1}-2\right\} \quad \ldots \quad \ldots \quad (21)$$

$$\frac{Z'(\eta)}{Z(\eta)} = \frac{1}{a\eta}\left[\frac{4\pi\,\Omega(\eta)}{Z(\eta)}-3\right] \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

$$-\frac{1}{N\eta_0^{1/b}}\left\{\frac{\Omega(\eta)}{\beta}\right\}^{\frac{1}{2}}\frac{\Upsilon(\eta)}{P(\eta)} + G\,Z(\eta) - U(\eta)\left\{1-U(\eta)\right\} + \frac{P(\eta)}{\Omega(\eta)}$$

$$U'(\eta) = \frac{\left\{\dfrac{a\{2-3U(\eta)\}}{b+aU(\eta)}\right\}}{\left[\dfrac{a^2}{(b+aU(\eta))}\cdot\dfrac{P(\eta)}{\Omega(\eta)}-\left\{b+aU(\eta)\right\}\right]\eta} \quad \ldots \quad \ldots \quad (23)$$

where

$$N = \frac{4\,a_1\,c\,\mu_0}{3\,\beta^{1/2}\Gamma^2}\cdot\frac{1}{\eta_0^{1/b}}\,. \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

The appropriate shock conditions are

$$U(\eta_0) = \frac{b}{a}\left[\frac{1}{\gamma M^2}-1\right] \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

$$\frac{\Omega(\eta_0)}{\beta} = \eta_0^{2/b}\cdot\gamma M^2 \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

$$\frac{P(\eta_0)}{\beta} = -\frac{4\pi G\beta(\eta_0)^{4/b}}{\left(2\dfrac{a}{b}+3\right)\left(4\dfrac{a}{b}+2\right)}\cdot\gamma M^2 \quad \ldots \quad (27)$$

$$\frac{\Upsilon(\eta_0)}{\beta} = -\frac{1}{2}\cdot\frac{b^3}{a^3}\,\eta_0^{2/b}\left[\frac{1}{\gamma^2 M^4}-1\right] \quad \ldots \quad (28)$$

$$\frac{Z(\eta_0)}{\beta} = \frac{4\pi}{2\dfrac{a}{b}+3}\cdot(\eta_0)^{2/b} \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

where

$$M^2 = \frac{-\frac{b^2}{a^2}\left(\frac{2a}{b}+3\right)\left(\frac{4a}{b}+2\right)}{4\pi\, G\gamma\beta} \cdot \frac{1}{(\eta_0)^{2/b}} \quad \ldots \quad \ldots \quad (30)$$

It should be observed that the constants a and b are still open. The above equations and conditions represent the problem of an explosion, where the total energy of the disturbance is proportion as some power of time. They include the two special cases of interest, namely, constancy of the total energy of the wave and constancy of the shock velocity.

The two characteristic parameters, representing the strength of the explosion and properties of the medium controlling radiation are M [eqn. (30)] and $\mathcal{N}$ [eqn. (24)] respectively.

4. Numerical Results

For the purpose of numerical integrations we consider the case of constant total energy of disturbance and this implies $a = -5$ and $b = 4$.

The ordinary differential equations (19)—(23), subject to the external boundary conditions (25)—(29), are integrated, following an extension of the Runge-Kutta method (Scarborough 1955) for two such equations.

Numerical results for certain choice of parameters are represented in Tables I—III. At the shock surface, without any loss of generality, we take $\eta_0 = 1$. Solutions are continued till we arrive at the singular surface $b+a\ U(\eta) = 0$, marking the interface of the exploding mass and the expanding region of the gaseous configuration.

Table I

$\gamma = 5/3,\ \mathcal{N} = 10,\ M^2 = 20$

η	$U(\eta)$	$\frac{\Omega(\eta)}{\beta}$	$\frac{P(\eta)}{\beta}$	$\frac{\Upsilon(\eta)}{\beta}$	$\frac{\mathcal{Z}(\eta)}{\beta}$
1	0·7760	33·3333	0·6400	—0·2557	25·1428
1·02	0·7779	34·2540	0·6319	—0·2589	23·7610
1·04	0·7799	35·2807	0·6220	—0·2619	22·3329
1·06	0·7819	36·4263	0·6114	—0·2650	20·8639
1·08	0·7838	37·7263	0·5999	—0·2682	19·3484
1·10	0·7857	39·2230	0·5875	—0·2714	17·7524
1·14	0·7895	43·1052	0·5593	—0·2779	14·4137
1·18	0·7932	49·0158	0·5252	—0·2844	10·7054
1·22	0·7968	59·8392	0·4815	—0·2909	6·3766
1·256	0·7997	—	—	—	—

TABLE II

$\gamma=5/3,\ N=100,\ M^2=20$

η	$U(\eta)$	$\frac{\Omega(\eta)}{\beta}$	$\frac{P(\eta)}{\beta}$	$\frac{\Upsilon(\eta)}{\beta}$	$\frac{Z(\eta)}{\beta}$
1	0·776	33·3333	0·6400	—0·2558	25·1429
1·02	0·7769	32·7280	0·6303	—0·2572	23·7896
1·04	0·7778	32·1424	0·6207	—0·2587	22·4760
1·06	0·7787	31·5567	0·6111	—0·2602	21·2008
1·08	0·7796	30·9810	0·6015	—0·2618	19·9612
1·10	0·7805	30·4153	0·5921	—0·2635	18·7583
1·14	0·7823	29·3138	0·5735	—0·2669	16·4545
1·18	0·7841	28·2622	0·5554	—0·2705	14·2773
1·22	0·7860	27·2535	0·5378	—0·2743	12·2164
1·30	0·7898	25·3641	0·5042	—0·2825	8·4079
1·38	0·7937	23·6417	0·4729	—0·2913	4·9702
1·50	0·8000	21·3140	0·4299	—0·3056	0·3903

TABLE III

$\gamma=4/3,\ M^2=20,\ N=100$

η	$U(\eta)$	$\frac{\Omega(\eta)}{\beta}$	$\frac{P(\eta)}{\beta}$	$\frac{\Upsilon(\eta)}{\beta}$	$\frac{Z(\eta)}{\beta}$
1	0·88	26·6667	0·6400	—0·2556	25·1328
1·02	0·7719	27·2124	0·6355	—0·2611	24·0845
1·04	0·7738	28·2724	0·6305	—0·2666	23·0168
1·06	0·7757	28·9414	0·6248	—0·2721	21·9042
1·08	0·7776	29·6709	0·6186	—0·2776	20·7670
1·12	0·7812	31·3505	0·6040	—0·2885	18·4088
1·16	0·7848	32·9803	0·5874	—0·2993	15·9168
1·2	0·7883	35·6151	0·5415	—0·3100	13·2980
1·24	0·7918	39·8303	0·5117	—0·3208	10·4361
1·28	0·7952	46·1889	0·4877	—0·3312	7·1844
1·346	0·7995	72·7858	0·4250	—0·3448	1·3951

5. Discussions

Assumption of isothermal shock conditions involves solutions which are markedly in deviation from the corresponding behaviour in the absence of any radiation. This one can readily observe from comparison with the known solutions in the reference already provided in the introduction.

Further, the solutions terminate very near the shock surface for higher values of the Mach number and so are not reproduced. Although not included, similar is the case, even when $M^2=20$, for an isothermal medium where the shock velocity for such types of flow remains constant.

Besides, we find that the radiation flux is negative in the entire region and increases in magnitude as the centre is approached. This is as expected, showing that there is more of absorption of radiation than emission as the expanding gaseous elements are more and more heated up by radiation towards the centre.

It will be interesting if, one could compare the results with observational values.

The solutions appear to yield tenable results in the initial stages of a not too strong central explosion for the model considered.

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