BOOLEAN COMPLETIONS AS INJECTIVE HULLS OF MODULES OVER BOOLEAN RINGS

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In this paper injective hulls of some modules over a Boolean ring R have been described as Boolean completions. The following results have been obtained:
(1) Suppose S is a Boolean ring and also an R-module. If $I \subseteq R$ is an ideal and $f : I \to S$ is an R-module homomorphism such that $V f(I)$ exists in S, then there exists an element c in S such that $f(x) = xc$ for all x in I. (2) If $J \subseteq R$ is an ideal and $Q \supseteq R/J$ a Boolean completion of R/J as a Boolean ring, then Q can be made into an R-module and as such it is an R-injective hull of R/J. (3) Let $I \subseteq R$ be an ideal, $\bar{R}$ a Boolean completion of R as a Boolean ring and $\bar{I} = \{V S \mid S \subseteq I\}$. Then $\bar{I}$ is a complete ideal of R and as an $\bar{R}$-module, it is an injective hull of I. (4) Let S be any commutative ring with unit and $I \subseteq S$ an ideal such that I is a Boolean ring (unit not assumed in I). If I is S-injective, then I has the unit element and is I-injective. (5) An ideal $I \subseteq R$ is injective iff it is complete as a Boolean ring. (6) An ideal $J \subseteq R$ is normal iff $\bar{J} \cap R = J$. (7) If $J \subseteq R$ is a normal ideal, then $\bar{R}/\bar{J}$ is an R-injective hull of R/J. Corollary—If $J \subseteq R$ is a complete ideal, then $\bar{R}/\bar{J}$ is an R-injective hull of R/J.

INTRODUCTION

A construction of an injective hull of a semi-simple module over a commutative regular ring, which is more general than a Boolean ring, has been given by Tiwary (1969). The object of this paper is to give a description of the injective hulls of cyclic modules over a Boolean ring R and also of ideals considered as R-modules. The main tool in this construction is the concept of Boolean completion. It turns out that the R-injective ideals are precisely those ideals which are complete as Boolean rings. In the case of normal ideals we obtain the interesting result that the completion of a quotient of R by a normal ideal is the quotient of their completions.

DEFINITIONS AND NOTATIONS

Throughout our discussion R will denote a Boolean ring with unit e. It is well known (Birkoff 1948; Doctor 1964) that the divisibility relation $\leqslant$ in R defined by $r \leqslant s$ iff $rs = r$ is a partial ordering and with respect to this

relation R is a Boolean lattice where meet ($\wedge$), join ($\vee$) and complement ($'$) are given by $x \wedge y = xy$, $x \vee y = x+y+xy$ and $x' = x+e$ (Sikorski 1960). For any subset T of R, the symbol $V\,T$ or $V_{t\in T}\, t$ will denote the least upper bound of elements of T. This supremum need not always exist in R.

Definition 1—R is said to be 'complete' iff $V\,T$ exists in R for all $T \subseteq R$.

Definition 2—A Boolean ring S is called a 'Boolean completion' of R iff (Brainerd and Lambek 1959; Sikorski 1960)

(1) S contains R as a subring,
(2) S is a complete ring, and
(3) $s \in S$ implies $s = V\{r \in R \mid r \leqslant s\}$.

Definition 3—An ideal $J \subseteq R$ is a 'complete ideal' of R iff

$$A \subseteq J,\ V\,A \in R \text{ implies } V\,A \in J.$$

An ideal $J \subseteq R$ is said to be 'complete as a Boolean' ring iff for all $A \subseteq J$, $V\,A$ exists in J.

The following two lemmas are well-known facts about Boolean lattices (Halmos 1963; Sikorski 1960):

Lemma 1—Let $A \subseteq R$ such that $V\,A$ exists in R. Then for any $b \in R$, $b\,.\,VA$ exists in R and $b\,.\,V\,A = V\{ba \mid a \in A\}$.

Lemma 2—Let R be complete and $J \subseteq R$ an ideal which is complete as a Boolean ring. Then R/J is complete.

Lemma 3—Suppose S is a Boolean ring and also an R-module. If $I \subseteq R$ is an ideal and $f : I \to S$ is an R-module homomorphism such that $V\,f(I)$ exists in S, then there exists an element c in S such that $f(x) = xc$ for all x in I.

PROOF: Let $c = V\,f(I)$. Then $f(x) \leqslant c$ for any x in I. Hence $f(x) = f(x^2) = x\,f(x) \leqslant xc$. On the other hand, $xc = x\,V\{f(y) \mid y \in I\} = V\{x\,.f(y) \mid y \in I\}$ (by Lemma 1) $= V\{f(xy) \mid y \in I\}$. Now, for each $y \in I$, $f(xy) = y\,f(x) \leqslant f(x)$, hence $xc = V\{f(xy) \mid y \in I\} \leqslant f(x)$. This implies $xc = f(x)$ for all x in I as required.

The following theorem describes the injective hull of a cyclic R-module:

Theorem 1—If $J \subseteq R$ is an ideal and $Q \supseteq R/J$ a Boolean completion of R/J as a Boolean ring, then Q can be made into an R-module and as such it is an R-injective hull of R/J.

PROOF: Let $h : R \to R/J$ be the natural homomorphism, then Q being an R/J-module can be made into an R-module by $rx = h(r)x$ where $r \in R$, $x \in Q$. Let $I \subseteq R$ be any ideal and $f : I \to Q$ any R-module homomorphism. Since $c = V\,f(I)$ exists in Q, by Lemma 3 we have $f(x) = xc$ for all $x \in I$. The fact that R is with unit, then shows that Q is R-injective. To prove that Q is an R-essential extension of R/J, consider any non-zero element x in Q.

Since Q is a Boolean completion of R/J, we have $0 \neq x = V\{y \in R/J \mid y \leqslant x\}$. This implies that there exists a non-zero element $y \in R/J$ with $y \leqslant x$ (Cartan and Eilenberg 1956). Hence we have $0 \neq y = yx = h(r)x = rx$ for some $r \in h^{-1}(y)$. Therefore, $0 \neq y \in Rx \cap R/J$. This proves that R/J is a large submodule of Q. Thus Q is an R-injective hull ot R/J.

As an immediate consequence of the above theorem we have

Corollary 1—R/J is R-injective iff it is complete.

Corollary 2—R is injective iff it is complete (Brainerd and Lambek 1959).

PROOF: Take $J = 0$ in Corollary 1.

Remark 1: Since any Boolean completion of R as a Boolean ring is an injective hull of R, it follows that such a completion is unique up to isomorphism over R (Brainerd and Lambek 1959).

Remark 2: Corollary 2 should not be confused with the analogous, but a different theorem that R is injective in the category of Boolean rings and unitary ring homomorphisms iff it is complete (Halmos 1961).

Corollary 3—Any simple R-module is injective.

PROOF: Any simple module is isomorphic to R/P for some maximal ideal $P \subseteq R$ and R/P is isomorphic to $Z/Z2$, as is well known. But it is finite, hence complete and, therefore, injective.

Remark 3: Corollary 3 also follows from Rosenberg and Zelinsky (1959) since Boolean rings are commutative regular rings.

We now find an injective hull of an ideal of R considered as an R-module in the next:

Theorem 2—Let $I \subseteq R$ be an ideal, $\bar{R}$ a Boolean completion of R as a Boolean ring and $\bar{I} = \{V S \mid S \subseteq I\}$. Then $\bar{I}$ is a complete ideal of $\bar{R}$ and as an R-module it is an injective hull of I.

PROOF: Let $\bar{a}$ be any element in $\bar{I}$. Then $\bar{a}$ is the join of some elements in I and hence also the join of all its lower bounds in I. Thus

$$\bar{a} \in \bar{I} \text{ implies } \bar{a} = V\{a \in I \mid a \leqslant \bar{a}\}. \quad .. \quad .. \quad .. \quad (1)$$

Now, let $S = \{s_\alpha \mid \alpha \in A\} \subseteq \bar{I}$ and $\bar{s} = V S$.
Then $\bar{s} = V_\alpha s_\alpha = V_\alpha V_\beta t_{\alpha\beta}$ with $t_{\alpha\beta} \in I$, belongs to I. Thus

$$S \subseteq \bar{I} \text{ implies } V S \in \bar{I}. \quad .. \quad .. \quad .. \quad .. \quad (2)$$

Now, let $\bar{r} \in \bar{R}$, $\bar{a} \in \bar{I}$ with $\bar{r} = \underset{r \leqslant \bar{r}}{V} r$ and $\bar{a} = \underset{a \leqslant \bar{a}}{V} a$, then $\bar{r}\bar{a} = V\{ra \mid r \leqslant \bar{r}, a \leqslant \bar{a}\} \in \bar{I}$ since each $ra \in I$. Thus we have

$$\bar{r} \in \bar{R}, \bar{a} \in \bar{I} \text{ implies } \bar{r}\bar{a} \in \bar{I}. \quad .. \quad .. \quad .. \quad (3)$$

In particular,

$$\bar{a}\bar{b} \in \bar{I} \text{ whenever } \bar{a}, \bar{b} \text{ are in } \bar{I}. \quad .. \quad .. \quad .. \quad (4)$$

In order to prove that $\bar{I}$ is closed under addition, consider any two elements $\bar{a}$, $\bar{b}$ in $\bar{I}$. Then $(\bar{a}+\bar{b})(\bar{a}+\bar{b}+\bar{a}\bar{b}) = \bar{a}+\bar{b}$ shows that $\bar{a}+\bar{b} \leqslant \bar{a} \vee \bar{b} \in \bar{I}$. It follows that $\bar{a}+\bar{b} \in \bar{I}$ since $\bar{I}$ is a Boolean lattice ideal of $\bar{R}$ in view of (2) and (3). Hence

$$\bar{a}, \bar{b} \in \bar{I} \text{ implies } \bar{a}+\bar{b} \in \bar{I}. \quad .. \quad .. \quad .. \quad (5)$$

As a consequence of (2), (3), (4) and (5), it follows that $\bar{I}$ is an ideal of $\bar{R}$, is complete as a Boolean ring and is an R-module.

To show that $\bar{I}$ is R-injective, let $J \subseteq R$ be any ideal and $f: J \to \bar{I}$ any R-module homomorphism. Then since $\bar{V} f(J)$ exists in $\bar{I}$, Lemma 3 implies the existence of an element $\bar{c} \in \bar{I}$ such that $f(x) = x\bar{c}$ for all $x \in J$. This proves the R-injectivity of $\bar{I}$ (Cartan and Eilenberg 1956).

Now, let $\bar{x}$ be any non-zero element in $\bar{I}$, $\bar{x} = \bar{V}\{x \in I \mid x \leqslant \bar{x}\}$. Then there exists $x \in I$ such that $0 \neq x \leqslant \bar{x}$. Therefore, $0 \neq x = x\bar{x} \in R\bar{x} \cap I$. Hence $\bar{I}$ is an R-essential extential extension of I and consequently an injective hull of I as an R-module.

Lemma 4—Let S be any commutative ring with unit and $I \subseteq S$ an ideal such that I is a Boolean ring (unit not assumed in I). If I is S-injective, then I has the unit element and is I-injective.

PROOF: Let $g: I \to I$ be the identity mapping. Then $g(sa) = sa = sg(a)$ for every $s \in S$, $a \in I$. Hence g is an S-homomorphism. Since I is injective as an S-module, this implies that there exists $a_0 \in I$ such that $a = g(a) = aa_0$ for all $a \in I$. Therefore, a_0 is the unit element in I. Now, let $J \subseteq I$ be an I-ideal then $ax = ax^2 = (ax)x$ belongs to J for any $a \in S$, $x \in J$, hence J is an S-ideal. Moreover, any I-homomorphism $f: J \to I$ is an S-homomorphism since $f(sb) = f(sb^2) = b\,sf(b) = sf(b^2) = sf(b)$ for any $b \in J$, $s \in S$. As a consequence of S-injectivity of I and the fact that S is with unit, there exists $r_0 \in I$ such that $f(b) = br_0$ for all $b \in J$. Since I contains unit, this implies that I is I-injective (Cartan and Eilenberg 1956).

A necessary and sufficient condition for the R-injectivity of an ideal of R is given in the following theorem:

Theorem 3—An ideal $I \subseteq R$ is injective iff it is complete as a Boolean ring.

PROOF: Let $\bar{I} = \left\{ \overset{\bar{R}}{\bar{V}} A \;\middle|\; A \subseteq I \right\}$ where $\bar{R}$ is a Boolean completion of R as a Boolean ring. Now if I is complete then $I = \bar{I}$, hence by Theorem 2, I is R-injective. Conversely if I is R-injective, then by Lemma 4, I is I-injective. Hence by Corollary 2 of Theorem 1, I is complete as a Boolean ring.

Corollary—For R, the following statements are equivalent:

(1) Each ideal $I \subseteq R$ is complete as a Boolean ring;
(2) R is semi-simple; and
(3) R is finite.

PROOF: (1) implies (2): Suppose (1) holds, then by Theorem 3, every ideal I of R is R-injective and hence a direct summand of R. Thus R is semi-simple. (2) implies (3): If R is semi-simple, it is a direct sum of finitely many simple components A_i $(i = 1, 2, \ldots, n)$.
Since each A_i is isomorphic to R/P_i where P_i is a maximal ideal of R, it is finite. Hence R is finite. (3) implies (1): If R is finite, then each ideal $I \subseteq R$ is finite and therefore complete as a Boolean ring.

Definition 4—For $A \subseteq R$, define $M_iA = \{x \in R \mid x \leqslant A\}$ and $M_aA = \{x \in R \mid x \geqslant A\}$. An ideal $J \subseteq R$ is called 'normal' iff $M_iM_aJ = J$. Since for any ideal J, $J \subseteq M_iM_aJ$ always holds, the condition $M_iM_aJ \subseteq J$ is sufficient to ensure that an ideal J is a normal ideal.

Let $\bar{R}$ be a Boolean completion of R as a Boolean ring. It was shown in Theorem 2 that if $J \subseteq R$ is any ideal, then $\bar{J} = \left\{ \overset{\bar{R}}{\vee} I \,\middle|\, I \subseteq J \right\}$ is a complete ideal of $\bar{R}$. The following lemma characterizes normal ideals of R.

Lemma 5—An ideal $J \subseteq R$ is normal iff $\bar{J} \cap R = J$.

PROOF: Let $c = \overset{\bar{R}}{\vee} J$. Then $\bar{J} = \bar{R}c$. Suppose J is a normal ideal of R and let r be any element in $\bar{J} \cap R$. Then $r \leqslant c \leqslant M_aJ$. It follows that $r \in M_iM_aJ = J$, hence $\bar{J} \cap R = J$. Conversely, let $\bar{J} \cap R = J$. The element c, being in $\bar{R}$, can be expressed as $c = \wedge B$ where $B = \{b \in R \mid c \leqslant b\}$. Hence $\bar{J} = \bar{R}c = \cap\{\bar{R}b \mid b \in B\}$. By the given condition we, therefore, have $J = R \cap \bigcap_{b \in B} \bar{R}b$. We want to show that $J = \bigcap_{b \in B} Rb$. Clearly $\bigcap_{b \in B} Rb \cap J$. To show the inclusion in the other direction, let $x \in J$. Then, since $x \leqslant c$, one has $x \leqslant b$ for all $b \in B$. Hence $x \in \bigcap_{b \in B} Rb$ so that $J = \bigcap_{b \in B} Rb$ as desired. If y be any element in M_iM_aJ, then $y \leqslant t$ for all $t \in R$ with $t \geqslant J$. Since every $b \in B$ satisfies the condition $b \in R$, $b \geqslant c \geqslant J$, we have $y \leqslant b$ for all b in B. Hence y belongs to $\bigcap_{b \in B} Rb = J$. This proves that $J \subseteq R$ is a normal ideal.

Theorem 4—If $J \subseteq R$ is a normal ideal, then $\bar{R}/\bar{J}$ is an R-injective hull of R/J.

PROOF: Consider the mapping $f: R/J \to \bar{R}/\bar{J}$ by $f(r+J) = r+J$. It is clearly an R-module homomorphism and is one-to-one since by Lemma 5, $\bar{J} \cap R = J$. Thus $\bar{R}/\bar{J}$ contains an isomorphic copy of R/J.

To prove that $\bar{R}/\bar{J}$ is an essential extension of R/J, let $\bar{r}+\bar{J}$ be a non-zero element in $\bar{R}/\bar{J}$. Hence $\bar{r} = V\{r \in R \mid r \leqslant \bar{r}\}$ lies outside J. We conclude that there exists $r \in R$ with $r \leqslant \bar{r}$, $r \notin J$; hence $r \notin \bar{J}$. Thus we have $0 \neq r+\bar{J} = r(\bar{r}+\bar{J})$ which shows that $f(R/J) \cap R(\bar{r}+\bar{J}) \neq 0$. Therefore, $\bar{R}/\bar{J}$ is an essential extension of $f(R/J)$. Consequently R/J is a large submodule of $\bar{R}/\bar{J}$.

As to the injectivity of $\bar{R}/\bar{J}$, we observe that $\bar{R}/\bar{J}$ is a complete Boolean ring and hence for any R-homomorphism φ from an ideal $I \subseteq R$ to $\bar{R}/\bar{J}$, $V\varphi(I)$ exists in $\bar{R}/\bar{J}$. We can therefore apply Lemma 3 to get the desired R-injectivity of $\bar{R}/\bar{J}$. Thus $\bar{R}/\bar{J}$ is an injective hull of R/J.

Corollary—If $J \subseteq R$ is a complete ideal, then $\bar{R}/\bar{J}$ is an R-injective hull of R/J.

PROOF: Take $\bar{a} \in \bar{J} \cap R$. Then $\bar{a} = V S$ for some $S \subseteq J$. Since J is a complete ideal of R and $\bar{a} = V S \in R$ we have $\bar{a} \in J$. Hence $\bar{J} \cap R \subseteq J$ and so J is a normal ideal of R. The result follows immediately from Theorem 4.

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