INTEGRAL EQUATION METHOD OF SOLVING TWO-DIMENSIONAL LAPLACE EQUATION—II

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In a previous paper (Bhargava and Saxena 1970), two methods were given to solve the two-dimensional Laplace equation ($\Delta^2\Phi = 0$) in a given bounded closed domain, when the values of the function Φ are given at the boundary of the domain. The example given therein was that of a circle, which has a constant curvature and is, therefore, comparatively easy to solve. In this paper the problem of a rectangle is solved using the general principles of the above methods. Because of the jump in the direction of the tangent on the boundary of the rectangle by an angle $\pi/2$ at its four corners, the methods are to be modified. The results are then obtained for two cases and are compared with the analytical results. The methods seem to be very encouraging.

INTRODUCTION

Laplace equation is an important partial differential equation, governing a large number of physical problems arising in many branches of physics, engineering and applied mathematics. It is generally solved under boundary conditions of three types (i) when the value of the function, or (ii) of its normal derivative, or (iii) of function on one part of the boundary and normal derivative on the remaining part is prescribed. They are usually called the Dirichlet, Neumann and Cauchy problems. It is proved (Kellog 1929) that under Dirichlet's condition the Laplace equation possesses a unique solution. In the case of the Neumann problem the solution is determined up to a constant. As mentioned (Kellog 1929) in the case of Neumann problem an additional condition is also to be fulfilled. The fact that certain Cauchy problems have unique solutions appears to have been established first by Lichtenstein (1913). Later efforts are also discussed by Miranda (1955). Analytical solution is known only for a few cases. For most of the problems one has to rely on approximate methods. The various methods available for this purpose are the following.

Iterative and relaxation methods based upon finite differences (Greenspan 1959), variational methods (Collatz 1960), Newton's method (Bellman

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et al. 1961), Synge's (1957) method of hypercircle, method of linear programming (Young 1962), method of Kernel functions (Nehari 1956), method of discrete Green's function (Berger and Lasher 1958), method of reduction to ordinary differential equations (Albrecht 1960), graphical methods (Panov 1934), boundary contraction method (Chow and Milnes 1960), Monte Carlo method (Ehrlich 1959) and approximate conformal mapping method (Warschawski 1956), etc. Only one reference is given for each of the above methods, other references may be found in the papers cited above.

Methods involving the potential of a single layer or potential of the double layer were proposed by Bhargava (1959) and then appeared later with some modifications (Bhargava and Radhakrishna 1963). Papers based upon these ideas have also lately appeared (Jaswon and Ponter 1963; Symm 1963). The singular character of the Kernel of the integral equations in these methods requires consideration. The methods for calculating the approximate value of the integral at these singular points are suggested for both the methods. To test the ideas, two problems whose analytical solutions are available are numerically solved. The results given in the tabular form indicate the accuracy of the methods for these problems.

It may be remarked here that the methods were applied earlier for the case of the circular boundary. This had the advantage of constant curvature and the formulae became pretty accurate (the error being one in a million). But for the rectangle, this is not true because of the jump in tangent by an angle $\pi/2$ at the four corners and the methods are to be modified.

Description of Methods

A harmonic function $V(P)$ at any point P in a domain D, bounded by a closed contour C may be represented by involving a single layer potential. Thus

$$V(P) = -\int_C \sigma(q) \log |q-P| \, dq \qquad \text{(1)}$$

where q is a boundary point on C and $|q-P|$ the distance between q and any point P. We distinguish a point on the boundary by the small letter p and denote the value of $V(P)$ at the boundary point p by $\Phi(p)$, thus

$$\Phi(p) = -\int_C \sigma(q) \log |q-p| \, dq. \qquad \text{(2)}$$

Hence to solve the Dirichlet problem one must find the value of $\sigma(q)$, called the density of the distribution, from (2) which is a Fredholm equation of the first kind and substitute in (1) to get finally the value of $V(P)$.

The same problem can be solved by introducing a double layer potential as follows:

$$W(P) = \frac{1}{2\pi}\int_C \mu(q) \log' |q-p| \, dq \qquad \text{(3)}$$

where the symbols have the same meaning as in (1) and dash denotes the outward drawn normal derivative of $\log|q-p|$ at q. This yields the boundary equation,

$$g(p) = \frac{1}{2\pi}\int_C \mu(q) \log' |q-p|\, dq + \tfrac{1}{2}\mu(p) \qquad \ldots \quad \ldots \quad (4)$$

where $W(P) \equiv g(p)$ on the boundary C. The term $\frac{1}{2}\mu(p)$ appears on the R.H.S. of (4) because of the jump in the normal derivative at p as it crosses the boundary.

Consequently, if $g(p)$ is given on the boundary, the value of $\mu(q)$, the density of the distribution, can be obtained from (4) which on being substituted in (3) provides $W(P)$.

Now onwards we shall call the former method as the First Method and the latter as Second Method. There is yet another form used by Bhargava (1959) and later the same formula was used by Jaswon and Ponter (1963) and Symm (1963) which involves the potentials of both layers, viz.:

$$w(P) = \frac{1}{2\pi}\int_C w'(q) \log|q-P|\, dq - \frac{1}{2\pi}\int_C w(q) \log'|q-P|\, dq \qquad \ldots \quad (5)$$

and the corresponding boundary equation is

$$\int_C w(q) \log'|q-p|\, dq + \pi w(p) = \int_C w'(q) \log|q-p|\, dq. \qquad \ldots \quad (6)$$

Thus, if $w(q)$ is given on the boundary, then we can find $w'(q)$ from (6), both of which on being substituted in (5) yields $w(P)$. Similarly, if $w'(q)$ is given (Neumann problem) on the boundary, one can obtain $w(P)$ with some complications because the coefficient matrix becomes singular involving arbitrariness in $w(P)$. Fredholm's equations of both kinds appear in these equations.

First Method

We indicate here how the First Method may be used for solving Dirichlet problem. Length of the contour may be divided into N equal intervals and assume that σ is constant on each of it. Then we may express (2) as

$$\phi(p) = -\int_C \sigma(q) \log|q-p|\, dq = -\sum_{k=1}^{N} \sigma(q_k) \int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log|q-p|\, dq$$

where q_k is the nodal point of the interval $I_k \equiv (q_{k-\frac{1}{2}}, q_{k+\frac{1}{2}})$, which is of length $2h$.

Replacing p by q_j, $j = 1, 2, \ldots, N$ and writing σ_k for $\sigma(q_k)$ for shortness in the above equation, we obtain

$$\phi(q_j) = -\sum_{k=1}^{N} \sigma_k \int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log|q-q_j|\, dq. \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

When $q \neq q_j$, integrals in (7) may be approximated by Simpson's rule as follows:

$$\int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log |q-q_j| \, dq = \frac{h}{3} [\log |q_{k+\frac{1}{2}}-q_j| + 4 \log |q_k-q_j| + \log |q_{k-\frac{1}{2}}-q_j|]. \qquad (8)$$

For $q = q_j$, the average value of $\log |q-q_j|$ is computed. This is done by approximating the arc length adjoining q_j to a straight line of length ϵ (taken as h), then the average value of $\log |q-q_j|$ is $(1/\epsilon) . \int_0^{\epsilon} \log S \, ds = \log \epsilon - 1$. Hence for this case,

$$\int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log |q-q_j| \, dq = 2h(\log \epsilon - 1). \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

Equation (7) represents a system of N-linear simultaneous equations, and it can be proved that its coefficient matrix is non-singular. Therefore, σ_k, $k = 1, 2, \ldots, N$ can be obtained by using any of the methods (Hildebrand 1956) for solving such a system, e.g. Gauss elimination method, etc. These values are to be substituted in (1) after writing it as

$$V(P) = -\sum_{k=1}^{N} \sigma_k \int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log |q-P| \, dq. \qquad \ldots \quad \ldots \quad (10)$$

It may be seen that none of the integrals in (10) is improper, and so can be easily computed by (8). Thus, we can compute finally $V(P)$.

Second Method

To solve the same problem by Second Method, one has to solve (4). Here, too, we divide the contour C into N equal intervals and assume μ to be constant on each of it. Thus

$$g(p) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu(q_k) \int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log' |q-p| \, dq + \tfrac{1}{2}\mu(p).$$

Putting $p = q_j$, $j = 1, 2, \ldots, N$ and $\mu(q_k) = \mu_k$ in the above equation

$$g(p) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log' |q-q_j| + \tfrac{1}{2}\mu_j. \qquad \ldots \quad \ldots \quad (11)$$

The integrand of the integrals in (11) can be evaluated as explained below:

$$\begin{aligned} \log' |q-q_j| &= \frac{\partial}{\partial \nu_q} \log |q-q_j| \\ &= \frac{\partial}{\partial x} \log |q-q_j| \frac{\partial x}{\partial \nu_q} + \frac{\partial}{\partial y} \log |q-q_j| \frac{\partial y}{\partial \nu_q} \\ &= \frac{(x-x_j) \cos \beta + (y-y_j) \sin \beta}{(x-x_j)^2 + (y-y_j)^2} \end{aligned}$$

where (x, y) and (x_j, y_j) are the coordinates of the points q and q_j respectively. The angle of inclination of the outward normal at q to x-axis is denoted by β.

There is yet another form by which $\log' |q-q_j|$ may be worked out. By Cauchy-Riemann equations, it is well known that $\log' |q-q_j| = \frac{\partial}{\partial \nu_q} \log |q-q_j| = \frac{\partial \theta}{\partial S}$, where θ is the angle which the radius vector $|q-q_j|$ makes with any fixed line in the plane, whence $\frac{\partial}{\partial \nu_q}|q-q_j|$ may be computed as the average change in θ as the point q moves a distance δS on C, the point q_j remains fixed. It is this approximation which is used in solving the above examples. Consequently, eqn. (11) turns out to be

$$f(q_j) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \int_{\theta_{k-\frac{1}{2}}}^{\theta_{k+\frac{1}{2}}} d\theta + \tfrac{1}{2}\mu_j \quad \ldots \quad (12)$$

where $\theta_{k-\frac{1}{2}}$ and $\theta_{k+\frac{1}{2}}$ are the inclinations of the radii vectors of the points $q_{k-\frac{1}{2}}$ and $q_{k+\frac{1}{2}}$ respectively from the point q_j with any fixed line.

Taking $\theta_k = (\theta_{k+\frac{1}{2}} - \theta_{k-\frac{1}{2}})$, we have

$$f(q_j) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \cdot \theta_k + \tfrac{1}{2}\mu_j$$

$$= \frac{1}{2\pi} \sum_{k \neq j} \mu_k \cdot \theta_k + \frac{1}{2\pi}(\theta_j + \pi)\mu_j. \quad \ldots \quad (13)$$

Since $g(q_j)$ is given, (13) represents a set of N-linear simultaneous equations for μ_k and it can be proved that its solution exists. Thus, μ_k, $k = 1, 2, \ldots,$ N may be obtained.

It may be observed that for each fixed q_j, $\sum_{j=1}^{N} \theta_j = \pi$, since the variable point q moves on C and covers its entire length (Fig. 1). It is one of the checks on the coefficients of (13) which is provided in this method. Having obtained μ_k, the value of $W(P)$ may be found as follows. Equation (3) may be put as

$$W(P) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu(q_k) \int_{q_{k-\frac{1}{2}}}^{q_{k+\frac{1}{2}}} \log' |q-P| \, dq$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \int_{\theta'_{k-\frac{1}{2}}}^{\theta'_{k+\frac{1}{2}}} d\theta = \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \cdot \theta'_k \quad \ldots \quad (14)$$

where θ'_k is the angle subtended by the interval I_k at the point P, which is inside the contour C (Fig. 2) and μ_k are the values obtained from (13).

To check the ideas, the contour C was taken as rectangle of sides 1 and 2 and the boundary values were chosen to be x (an odd function) and x^2-y^2 (an even function). The interior of C was covered by a square net of side 2, and the harmonic function was computed by both methods at each lattice

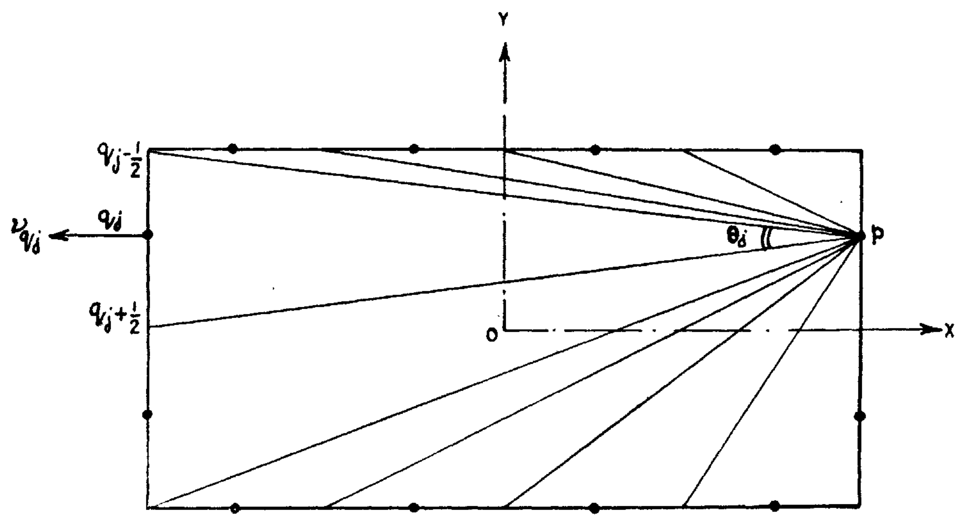

FIG. 1. Angles subtended by different intervals of the boundary at one of the nodal points.

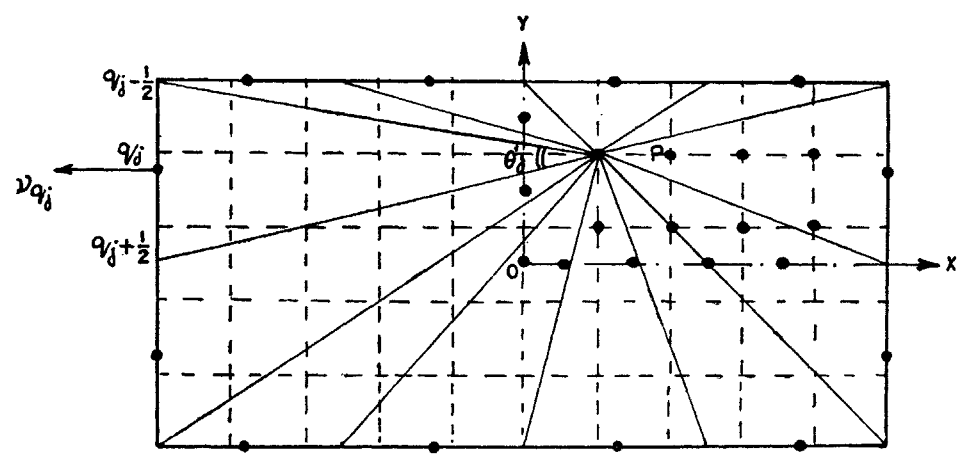

FIG. 2. Angles subtended by different intervals of the boundary at one of the lattice points of the net.

point in the positive quadrant (because of symmetry) and also at few points on both the axis (Fig. 2). The number of nodal points (N) on the boundary were successively taken to be 12, 24, 36 and 48. Results obtained in each example are given in Tables I–IV. Calculations were done on a Russian Computer MINSK-2 at the I.I.T., Bombay.

TABLE I

Values of W(P)

$g(p) = x$

Coordinates of the point P: X	Y	Analytic value	Computed value for $N = 12$	Absolute error	Computed error $N = 24$	Absolute error	Computed value for $N = 36$	Absolute error	Computed value for $N = 48$	Absolute error
0·20	0·10	0·2000000	0·2001989	0·0001989	0·1999252	0·0000748	0·1999644	0·0000356	0·1999791	0·0000209
0·40	0·10	0·4000000	0·3987845	0·0012155	0·3997860	0·0002140	0·3998960	0·0001040	0·3999390	0·0000610
0·60	0·10	0·6000000	0·5990817	0·0009183	0·5994801	0·0005199	0·5997444	0·0002556	0·5998501	0·0001499
0·80	0·10	0·8000000	0·7948987	0·0051013	0·7989135	0·0010865	0·7994470	0·0005530	0·7996733	0·0003267
0·20	0·30	0·2000000	0·2042370	0·0042370	0·1996848	0·0003152	0·2000107	0·0000107	0·1999959	0·0000041
0·40	0·30	0·4000000	0·3922933	0·0077067	0·3997494	0·0002506	0·3999835	0·0000165	0·3999856	0·0000144
0·60	0·30	0·6000000	0·6078152	0·0078152	0·5999812	0·0000188	0·5998924	0·0001076	0·5999430	0·0000570
0·80	0·30	0·8000000	0·7962643	0·0037357	0·7994848	0·0005152	0·7996189	0·0003811	0·7997794	0·0002206
0·30	0·00	0·3000000	0·2993992	0·0006008	0·2998570	0·0001430	0·2999299	0·0000701	0·2999589	0·0000411
0·50	0·00	0·5000000	0·4989167	0·0010833	0·4996327	0·0003673	0·4998203	0·0001797	0·4998946	0·0001054
0·70	0·00	0·7000000	0·6969975	0·0030025	0·6991619	0·0008381	0·6995959	0·0004041	0·6997635	0·0002365
0·10	0·00	0·1000000	0·1002416	0·0002416	0·0999636	0·0000364	0·0999821	0·0000179	0·0999895	0·0000105
0·00	0·20	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
4·00	0·40	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
0·00	0·00	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

TABLE II

Values of W(P)

$g(p) = x^2 - y^2$

Coordinates of the point P: X	Y	Analytic value	Computed value for $N = 12$	Absolute error	Computed value for $N = 24$	Absolute error	Computed value for $N = 36$	Absolute error	Computed value for $N = 48$	Absolute error
0·20	0·10	0·0300000	0·0408493	0·0108493	0·0328310	0·0028310	0·0312621	0·0012621	0·0307111	0·0007111
0·40	0·10	0·1500000	0·1582922	0·0082922	0·1521071	0·0021071	0·1509336	0·0009336	0·1505243	0·0005243
0·60	0·10	0·3500000	0·3533512	0·0033512	0·3504851	0·0004851	0·3501845	0·0001845	0·3500960	0·0000960
0·80	0·10	0·6299999	0·6142700	0·0157299	0·6275289	0·0024710	0·6287405	0·0012594	0·6292569	0·0007430
0·20	0·30	−0·0500000	−0·0391421	0·0108579	−0·0467821	0·0032179	−0·0485106	0·0014894	−0·0491668	0·0008332
0·40	0·30	0·0700000	0·0760629	0·0060629	0·0726131	0·0026131	0·0713283	0·0013283	0·0707506	0·0007506
0·60	0·30	0·2700000	0·2929424	0·0229424	0·2721476	0·0021476	0·2708980	0·0008980	0·2705220	0·0005220
0·80	0·30	0·5500000	0·5465955	0·0034045	0·5495888	0·0004112	0·5496353	0·0003647	0·5497867	0·0002133
0·30	0·00	0·0900000	0·0994060	0·0094060	0·0924680	0·0024680	0·0910964	0·0010964	0·0906169	0·0006169
0·50	0·00	0·2500000	0·2558478	0·0058478	0·2512962	0·0012962	0·2505590	0·0005590	0·2503101	0·0003101
0·70	0·00	0·4900000	0·4835933	0·0064067	0·4888882	0·0011118	0·4894629	0·0005371	0·4896826	0·0003174
0·10	0·00	0·0100000	0·0219212	0·0119212	0·0129292	0·0029292	0·0113066	0·0013066	0·0107363	0·0007363
0·00	0·20	−0·0400000	−0·0252918	0·0147082	−0·0367999	0·0032001	−0·0385803	0·0014197	−0·0391997	0·0008003
0·00	0·40	−0·1600000	−0·1318805	0·0281195	−0·1549289	0·0050711	−0·1581259	0·0018741	−0·1590289	0·0009711
0·00	0·00	0·0000000	0·0123823	0·0123823	0·0029821	0·0029821	0·0013302	0·0013302	0·0007497	0·0007497

TABLE III

Values of V(P)

$\Phi(p) = x$

Coordinates of the point P: X	Y	Analytic value	Computed value for $N = 12$	Absolute error	Computed value for $N = 24$	Absolute error	Computed value for $N = 36$	Absolute error	Computed value for $N = 48$	Absolute error
0·20	0·10	0·2000000	0·2017052	0·0017052	0·2002302	0·0002302	0·2000654	0·0000654	0·2000209	0·0000209
0·40	0·10	0·4000000	0·4031836	0·0031836	0·4005315	0·0005315	0·4001550	0·0001550	0·4000527	0·0000527
0·60	0·10	0·6000000	0·6042670	0·0042670	0·6009154	0·0009154	0·6002730	0·0002730	0·6000965	0·0000965
0·80	0·10	0·8000000	0·8049704	0·0049704	0·8012003	0·0012003	0·8002944	0·0002944	0·8000879	0·0000879
0·20	0·30	0·2000000	0·2020482	0·0020482	0·2001356	0·0001356	0·2000405	0·0000405	0·2000093	0·0000093
0·40	0·30	0·4000000	0·4034251	0·0034251	0·4004088	0·0004088	0·4001141	0·0001141	0·4000335	0·0000335
0·60	0·30	0·6000000	0·6051185	0·0051185	0·6011082	0·0011082	0·6003017	0·0003017	0·6001123	0·0001123
0·80	0·30	0·8000000	0·7997985	0·0002015	0·8017070	0·0017070	0·8008194	0·0008194	0·8003736	0·0003736
0·30	0·00	0·3000000	0·3024430	0·0024430	0·3003795	0·0003795	0·3001097	0·0001097	0·3000365	0·0000365
0·50	0·00	0·5000000	0·5037303	0·0037303	0·5007174	0·0007174	0·5002116	0·0002116	0·5000734	0·0000734
0·70	0·00	0·7000000	0·7045231	0·0045231	0·7009673	0·0009673	0·7002797	0·0002797	0·7000920	0·0000920
0·10	0·00	0·1000000	0·1008534	0·0008534	0·1001151	0·0001151	0·1000326	0·0000326	0·1000104	0·0000104
0·00	0·20	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
0·00	0·40	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000
0·00	0·00	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000	0·0000000

TABLE IV

Values of V(P)

$\Phi(p) = x^2 - y^2$

Coordinates of the point P: X	Coordinates of the point P: Y	Analytic value	Computed value for $N = 12$	Absolute error	Computed value for $N = 24$	Absolute error	Computed value for $N = 36$	Absolute error	Computed value for $N = 48$	Absolute error
0·20	0·10	0·0300000	0·0403123	0·0103123	0·0315811	0·0015811	0·0304634	0·0004634	0·0301605	0·0001605
0·40	0·10	0·1500000	0·1627852	0·0127852	0·1521549	0·0021549	0·1506438	0·0006438	0·1502282	0·0002288
0·60	0·10	0·3500000	0·3635667	0·0135667	0·3530054	0·0030054	0·3509070	0·0009070	0·3503209	0·0003209
0·80	0·10	0·6299999	0·6409821	0·0109821	0·6335260	0·0035260	0·6308293	0·0008293	0·6302171	0·0002172
0·20	0·30	−0·0500000	−0·0416322	0·0083678	−0·0488281	0·0011719	−0·0496602	0·0003398	−0·0498867	0·0001133
0·40	0·30	0·0700000	0·0846838	0·0146838	0·0717231	0·0017231	0·0705194	0·0005194	0·0701812	0·0001812
0·60	0·30	0·2700000	0·2880346	0·0180346	0·2736424	0·0036424	0·2710459	0·0010459	0·2704060	0·0004060
0·80	0·30	0·5500000	0·5505355	0·0005355	0·5551527	0·0051527	0·5525136	0·0025136	0·5511497	0·0011497
0·30	0·00	0·0900000	0·1015280	0·0115280	0·0918639	0·0018639	0·0905522	0·0005522	0·0901936	0·0001936
0·50	0·00	0·2500000	0·2632037	0·0132037	0·2525625	0·0025625	0·2507681	0·0007681	0·2502710	0·0002710
0·70	0·00	0·4900000	0·5018075	0·0118075	0·4929498	0·0029498	0·4908546	0·0008546	0·4902688	0·0002688
0·10	0·00	0·0100000	0·0197478	0·0097478	0·0114897	0·0014897	0·0104345	0·0004345	0·0101496	0·0001496
0·00	0·20	−0·0400000	−0·0316131	0·0083869	−0·0387374	0·0012626	−0·0396373	0·0003627	−0·0398772	0·0001228
0·00	0·40	−0·1600000	−0·1550040	0·0049960	−0·1593390	0·0006610	−0·1598330	0·0001670	−0·1599453	0·0000547
0·00	0·00	0·0000000	0·0094877	0·0094877	0·0014437	0·0014437	0·0042000	0·0004200	0·0001442	0·0001442

It appears that the Second Method is more consistent and the maximum error at any point of the net in both the examples is less than that in the First Method.

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