A NOTE ON IMMERSED MANIFOLDS

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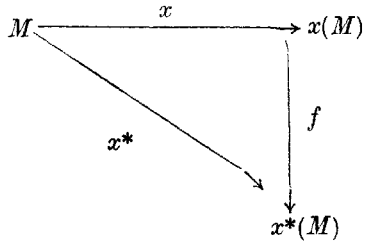
(Communicated by R. S. Varma, F.N.A.)

(*Received 15 December 1969*)

A diffeomorphism between two complete Riemannian manifolds of even dimension which are immersed in a Euclidean space has been discussed under certain assumptions.

Let M be an $n\,(=2p)$-dimensional complete Riemannian manifold of class c^2 and let it be immersed in an $(n+m)$-dimensional Euclidean space E^{n+m}, that is there exists a c^2-mapping $x: M \to E^{n+m}$, such that the induced linear mapping x_* on the tangent spaces is univalent everywhere. Let x and x^* be two such immersions of M in the Euclidean space E^{n+m}. Then there exists a diffeomorphism $f: x(M) \to x^*(M)$ such that the following diagram is commutative:



Let G and G^* be the fundamental tensors of the induced metrics defined on $x(M)$ and $x^*(M)$ respectively. Let $(e_1, \ldots, e_n)$ be a unit frame in M. Clearly, there is no loss of generality in identifying (e_i) with $x_*(e_i)$, let $(\omega_1, \omega_2, \ldots, \omega_n)$ be the co-frame dual to $(e_1, e_2, \ldots, e_n)$. Let $(e_{n+1}, e_{n+2}, \ldots, e_{n+m})$ be a frame in the normal bundle at $x(p)$ where p belongs to M. Then the structure equations are given by the following:

$$dx = \Sigma\, \omega_i e_i \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots \quad (1.1)$$

$$de_i = \Sigma\, \omega_{ik} e_k + \Sigma\, \omega_{i,\, n+s} e_{n+s} \qquad \ldots\ \ldots\ \ldots \quad (1.2)$$

where the indices $i, j, k \ldots = 1, 2, \ldots, n$ and s take the values from 1 to m. The forms ω_{ik} and $\omega_{i,\, n+s}$ are linear differential forms in the bundle induced over M from the principal fibre bundle of E^{n+m}. The mapping $f: x(M) \to x^*(M)$ is called an almost isometry if the induced linear mapping of

f maps the connection form defined in $x^*(M)$ onto the connection form defined in $x(M)$ (Chern and Hsuing 1963).

Exterior differentiation of (1.1) gives

$$d\omega_i = \omega_j \wedge \omega_{ji}$$

where we have used (1.2) and the fact that $\omega_i \wedge \omega_{i,\, n+s} = 0$. Also

$$de_{n+s} = \Sigma\, \omega_{n+s,\, j} e_j \quad .. \quad .. \quad .. \quad .. \quad (1.3)$$

$$d\omega_{i,\, n+s} = \Sigma\, \omega_{ij} \wedge \omega_{j,\, n+s}\,. \quad .. \quad .. \quad .. \quad (1.4)$$

Taking the exterior derivative of (1.2) and using (1.3) and (1.4), we obtain

$$\Omega_{ij} = \omega_{il} \wedge \omega_{jl} + d\omega_{ij} \quad .. \quad .. \quad .. \quad (1.5)$$

so that Ω_{ij} is the curvature form of the manifold.

Let u_i be the components of a unit vector v with respect to the basis (e_i) in the submanifold $x(M)$, so that

$$v = u_i e_i\,.$$

Therefore

$$dv = \theta_i e_i + u_i \omega_{i,\, n+s} e_{n+s}$$

where

$$\theta_i = du_i - \omega_{ij} u_j\,.$$

Take now the forms ϕ_k and ψ_k of degree $(2p-1)$ and $(2p)$ respectively which are defined by (Chern 1944)

$$\phi_k = \epsilon_{i_1 i_2 \ldots i_{2p}} u_{i_1} \theta_{i_2} \wedge \theta_{i_3} \wedge \ldots \wedge \theta_{i_{2p-2k}} \wedge \Omega_{i_{2p-2k+1},\, i_{2p-2k+2}} \wedge \ldots \wedge \Omega_{i_{2p-1},\, i_{2p}}$$

$$\text{for } k = 0,\, 1,\, \ldots (p-1)$$

$$\psi_k = \epsilon_{i_1 i_2 \ldots i_{2p}} \Omega_{i_1 i_2} \wedge \theta_{i_3} \wedge \ldots \wedge \theta_{i_{2p-2k}} \wedge \Omega_{i_{2p-2k+1},\, i_{2p-2k+2}} \wedge \ldots \wedge \Omega_{i_{2p-1},\, i_{2p}}$$

$$\text{for } k = 0,\, 1,\, \ldots (p-1)$$

In this case the curvature form Ω_{ij} is defined by (1.5). It can be proved after some calculations that

$$d\phi_{p-1} = \psi_{p-2} + \frac{1}{2p} \psi_{p-1}\,. \quad .. \quad .. \quad .. \quad (1.6)$$

To the form ϕ_{p-1}, we shall apply a special Stokes' theorem proved by Gaffney (1954) which states that:

Theorem—Let M be an orientable complete Riemannian manifold of class c^2, and let γ be an $(n-1)$ form of class c^1 defined in a region L with the property that both γ and $d\gamma$ are in L, then

$$\int_M d\gamma = 0.$$

Thus we have

$$\int_{x(M)} d\phi_{p-1} = 0.$$

Using (1.6) in the above equation, we have

$$\int_{x(M)} \psi_{p-2} + \frac{1}{2p} \int_{x(M)} \psi_{p-1} = 0$$

i.e.

$$\int_{x(M)} \epsilon_{i_1 \ldots i_{2p}} \Omega_{i_1 i_2} \wedge \theta_{i_3} \wedge \theta_{i_4} \wedge \Omega_{i_5 i_6} \wedge \ldots \wedge \Omega_{i_{2p-1}, i_{2p}}$$

$$+ \frac{1}{2p} \int_{x(M)} \epsilon_{i_1 \ldots i_{2p}} \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2p-1}, i_{2p}} = 0 \quad .. \quad .. \quad (1.7)$$

or

$$\int_{x(M)} \theta_{i_1} \wedge \theta_{i_2} \wedge \theta_{I(2p-2)} + (-1)^{p-1} \cdot 2^{2p-1} \pi^p (p-1)! \, \chi = 0$$

where χ is the Euler's Poincaré characteristic of the manifold and $\theta_{I(2p-2)}$ is $(2p-2)$th Gauss-curvature form defined by Hsuing and Yuk-Keung (1967).

Let u_i^*, Ω_{ij}^* be the components of a unit vector and curvature form of the manifold $x^*(M)$ and let $\theta_i^* = du_i^* - \omega_{ij}^* u_j^*$.

Similarly, we can introduce the forms ϕ_k^* and ψ_k^* of degree $(2p-1)$ and $(2p)$ respectively in the manifold $x^*(M)$. Also the diffeomorphism $f : x(M) \to x^*(M)$ induces a linear transformation f^* on the forms in $x^*(M)$ to forms in $x(M)$, i.e. $f^*(\phi_k^*)$ and $f^*(\psi_k^*)$ are forms in $x(M)$. Again applying the special Stokes' theorem of Mathew P. Gaffney to the form $f^*(\phi_{p-1}^*)$, we have

$$\int_{x(M)} f^*(\psi_{p-2}^*) + \tfrac{1}{2} \int_{x(M)} f^*(\psi_{p-1}^*) = 0$$

i.e.

$$\int_{x(M)} f^*(\theta_{i_1}^*) \wedge f^*(\theta_{i_2}^*) \wedge f^*(\theta_{I(2p-2)}^*) + (-1)^{p-1} \cdot 2^{2p-1} \pi^p (p-1)! \, \chi = 0$$

$$.. \quad (1.8)$$

where $\theta_{I(2p-2)}^*$ is $(2p-2)$th Gauss-curvature form of the submanifold $x^*(M)$.

Subtracting (1.8) from (1.7) we have

$$\int_{x(M)} \theta_{i_1} \wedge \theta_{i_2} \wedge \theta_{I(2p-2)} = \int_{x(M)} f^*(\theta_{i_1}^*) \wedge f^*(\theta_{i_2}^*) \wedge f^*(\theta_{I(2p-2)}^*).$$

Let us now assume that $(2p-2)$th Gauss-curvature form is invariant under the induced linear mapping of 'f', so that we have

$$\int_{x(M)} [\theta_{i_1} \wedge \theta_{i_2} - f^*(\theta_{i_1}^*) \wedge f^*(\theta_{i_2}^*)] \theta_{I(2p-2)} = 0.$$

Thus we have

$$\theta_{i_1} \wedge \theta_{i_2} = f^*(\theta_{i_1}^*) \wedge f^*(\theta_{i_2}^*).$$

From which we see that f is an almost isometry. Hence we obtain the following:

Theorem—Let $x(M)$ and $x^*(M)$ be two $(2p)$-dimensional complete Riemannian manifolds of class c^2 immersed in a Euclidean space E^{2p+m} such that $f : x(M) \to x^*(M)$ is a diffeomorphism. If $(2p-2)$th Gauss-curvature form is invariant under the induced linear mapping of f then f is an almost isometry.

References

Chern, S. S. (1944). A simple intrinsic proof of the Gauss-Bonnet formula. *Ann. Math.*, **45**, 747–52.

Chern, S. S., and Hsuing, C. C. (1963). On the isometry of compact submanifolds in Euclidean space. *Math. Ann.*, **149**, 278–85.

Gaffney, Mathew P. (1954). A special Stokes' theorem for complete Riemannian manifolds. *Ann. Math.*, **60**, 140–45.

Hsuing, C. C., and Yuk-Keung (1967). Curvature and characteristic classes of a compact Riemannian manifold. *J. Differential Geometry*, **1**, 89–97.