ON ALMOST HERMITE SPACES—NIJENHUIS TENSOR

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In this paper we have given various forms of Nijenhuis tensor and obtained some results concerning Kähler and almost Tachibana spaces.

1. INTRODUCTION

We consider a $2n$-dimensional space M_{2n} of differentiability class C^{r+1}. Let there be defined in M_{2n} a vector valued linear function F such that

$$\bar{\bar{X}}+X = 0 \qquad \text{(1.1)}$$

for arbitrary vector field X, where

$$\bar{X} \overset{\text{def}}{=} F(X). \qquad \text{(1.2)}$$

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an 'almost complex space'.

Agreement 1.1—All the equations which follow hold for arbitrary vector fields X, Y, Z, . . . , etc.

Let the almost complex space M_{2n} be also endowed with the Hermitian metric tensor g:

$$g(\bar{X}, \bar{Y}) = g(X, Y). \qquad \text{(1.3)}$$

Then M_{2n} is called an almost Hermite space.

Let us put

$$'F(X, Y) \overset{\text{def}}{=} g(\bar{X}, Y). \qquad \text{(1.4)}$$

Then from (1.1), (1.2), (1.3) and (1.4), we have

$$'F(\bar{X}, \bar{Y}) = -g(X, \bar{Y}) = g(\bar{X}, Y) = {}'F(X, Y) \qquad \text{(1.5a)}$$

$$-'F(\bar{X}, Y) = g(X, Y) = {}'F(X, \bar{Y}) \qquad \text{(1.5b)}$$

$$'F(X, Y)+{}'F(Y, X) = 0. \qquad \text{(1.6)}$$

Suppose D is a Riemannian connexion in M_{2n} (Mishra 1965):

$$D_XY-D_YX = [X, Y] \qquad \text{(1.7)}$$

$$X\,.\{g(Y, Z)\} = g(D_XY, Z)+g(Y, D_XZ). \qquad \text{(1.8)}$$

If in addition to (1.2), (1.3) and (1.6), we also have

$$(D_XF)(Y) = 0 \qquad \text{(1.9)}$$

M_{2n} is called Kähler space. An almost Hermite space for which

$$(D_X{}'F)(Y, Z)+(D_Y{}'F)(X, Z) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1.10)$$

is satisfied, is said to be an almost Tachibana space. An almost Hermite space for which

$$(D_X{}'F)(Y, Z)+(D_Y{}'F)(Z, X)+(D_Z{}'F)(X, Y) = 0 \quad \ldots \quad \ldots \quad (1.11)$$

is satisfied, is said to be an almost Kähler space. An almost Hermite space for which

$$\text{div } F = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.12)$$

is satisfied, is said to be an almost semi-Kähler space.

Nijenhuis tensor N is given by (Yano 1965; Mishra 1969)

$$N(X, Y) = [\bar{X}, \bar{Y}]-\overline{[X, \bar{Y}]}-\overline{[\bar{X}, Y]}+\overline{\overline{[X, Y]}}. \quad \ldots \quad (1.13)$$

For an almost Hermite space, we have

$$N(X, Y) = [\bar{X}, \bar{Y}]-\overline{[X, \bar{Y}]}-\overline{[\bar{X}, Y]}-[X, Y] \quad \ldots \quad \ldots \quad (1.14)$$

in consequence of (1.1).

From (1.13) or (1.14), it is obvious that Nijenhuis tensor is skew-symmetric

$$N(X, Y) = -N(Y, X). \quad \ldots \quad \ldots \quad \ldots \quad (1.15)$$

If we put

$$'N(X, Y, Z) \overset{\text{def}}{=} g(N(X, Y), Z) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.16)$$

$$'M(X, Y) \overset{\text{def}}{=} D_{\bar{X}}\bar{Y}-\overline{D_X\bar{Y}}-D_XY-\overline{D_{\bar{X}}Y} \quad \ldots \quad \ldots \quad \ldots \quad (1.17)$$

then

$$'M(X, Y, Z) = (D_{\bar{X}}{}'F)(Y, Z)+(D_X{}'F)(\bar{Y}, Z) \quad \ldots \quad \ldots \quad (1.18)$$

$$'N(X, Y, Z) = {}'M(X, Y, Z)-{}'M(Y, X, Z). \quad \ldots \quad \ldots \quad (1.19)$$

We have (Mishra 1969)

$$(D_X{}'F)(Y, \bar{Z}) = (D_X{}'F)(\bar{Y}, Z). \quad \ldots \quad \ldots \quad \ldots \quad (1.20)$$

2. Nijenhuis Tensor

Theorem 2.1—Let us put

$$B(X, Y) \overset{\text{def}}{=} \overline{[X, \bar{Y}]}+\overline{[\bar{X}, Y]}. \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

Then

$$B(\bar{X}, \bar{Y})+B(X, Y) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2a)$$

$$B(\bar{X}, Y) = B(X, \bar{Y}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2b)$$

$$\overline{B(\bar{X}, Y)} = -\{[\bar{X}, \bar{Y}]-[X, Y]\}. \quad \ldots \quad \ldots \quad (2.3)$$

Consequently,

$$N(X, Y) = -\overline{[B(\bar{X}, Y)}+B(X, Y)]. \quad \ldots \quad \ldots \quad (2.4)$$

Proof: Barring in (2.1) as indicated in (2.2), and using (1.1), we get (2.2) and (2.3). The equation (2.4) follows from (1.14), (2.1) and (2.2).

Theorem 2.2—Let us put

$$U(X, Y) \overset{\text{def}}{=} \overline{[\bar{X}, Y]} + [X, Y]. \quad \text{(2.5)}$$

Then

$$U(\bar{X}, \bar{Y}) = [\bar{X}, \bar{Y}] - \overline{[X, \bar{Y}]}. \quad \text{(2.6)}$$

Consequently,

$$N(X, Y) = U(\bar{X}, \bar{Y}) - U(X, Y). \quad \text{(2.7)}$$

Theorem 2.3—Put

$$V(X, Y) \overset{\text{def}}{=} \overline{[X, \bar{Y}]} + [X, Y]. \quad \text{(2.8)}$$

Then

$$V(\bar{X}, \bar{Y}) = [\bar{X}, \bar{Y}] - \overline{[\bar{X}, Y]}. \quad \text{(2.9)}$$

Consequently,

$$N(X, Y) = V(\bar{X}, \bar{Y}) - V(X, Y). \quad \text{(2.10)}$$

The proof of these two theorems follows the pattern of the proof of Theorem 2.1.

Note (2.1)—From (2.2), (2.3) and (2.4), it is obvious that

$$N(X, Y) = -N(\bar{X}, \bar{Y}) = \overline{N(X, \bar{Y})}. \quad \text{(2.11)}$$

Remark (2.1): If we put

$$'B(X, Y, Z) \overset{\text{def}}{=} g(B(X, Y), Z) \quad \text{(2.12a)}$$

$$'U(X, Y, Z) \overset{\text{def}}{=} g(U(X, Y), Z) \quad \text{(2.12b)}$$

$$'V(X, Y, Z) \overset{\text{def}}{=} g(V(X, Y), Z) \quad \text{(2.12c)}$$

then $'N(X, Y, Z)$ can be put in the form

$$'N(X, Y, Z) = 'B(\bar{X}, Y, \bar{Z}) - 'B(X, Y, Z) \quad \text{(2.13)}$$

$$'N(X, Y, Z) = 'U(\bar{X}, \bar{Y}, Z) - 'U(X, Y, Z) \quad \text{(2.14)}$$

$$'N(X, Y, Z) = 'V(\bar{X}, \bar{Y}, Z) - 'V(X, Y, Z). \quad \text{(2.15)}$$

Theorem 2.4—For a Kähler space, we have (Mishra 1969)

$$B(X, Y) = -\overline{B(\bar{X}, Y)} \quad \text{(2.16)}$$

$$U(\bar{X}, \bar{Y}) = U(X, Y) \quad \text{(2.17)}$$

$$V(\bar{X}, \bar{Y}) = V(X, Y). \quad \text{(2.18)}$$

Proof: For a Kähler space, we have

$$[X, \bar{Y}] = \overline{D_{\bar{Y}}\bar{X}} + \overline{D_X Y} \quad \text{(2.19)}$$

$$[\bar{X}, Y] = -\overline{D_Y X} - \overline{D_{\bar{X}}\bar{Y}}. \quad \text{(2.20)}$$

From (2.19), (2.20), (1.7) and (2.1), we obtain (2.16). Equations (2.17) and (2.18) can be obtained similarly.

Theorem 2.5—The necessary and sufficient condition for an almost Hermite space to be an almost Tachibana space is

$$'M(X, Y, Z) = (D_X{}'F)(Y, \bar{Z}) - (D_Y{}'F)(\bar{X}, Z) \quad \ldots \quad (2.21)$$

$$'N(X, Y, Z) = 2'M(X, Y, Z). \quad \ldots \quad (2.22)$$

PROOF: Barring X in (1.10) and using (1.20), we get

$$(D_{\bar{X}}{}'F)(Y, Z) + (D_Y{}'F)(X, \bar{Z}) = 0. \quad \ldots \quad (2.23)$$

Using this equation in (1.18), we obtain (2.21). (2.22) can be obtained from (1.19) and (2.21).

Theorem 2.6—The necessary and sufficient condition for an almost Hermite space to be an almost Tachibana space is

$$X\,.\,('F(Y, Z)) + Y\,.\,('F(X, Z)) = 'F(D_X Y, Z) + 'F(Y, D_X Z) + 'F(D_Y X, Z) + 'F(X, D_Y Z).$$

PROOF: We have

$$X\,.\,('F(Y, Z)) = (D_X{}'F)(Y, Z) + 'F(D_X Y, Z) + 'F(Y, D_X Z) \quad \ldots \quad (2.24)$$

Interchanging X, Y of this equation and adding the equation thus obtained with (2.24) and then using (1.10), we get the required result.

Theorem 2.7—For an almost Tachibana space, we have

$$3'M(X, Y, Z) = 2[(D_{\bar{X}}{}'F)(Y, Z) + (D_{\bar{Y}}{}'F)(Z, X) + (D_{\bar{Z}}{}'F)(X, Y)] \quad (2.25a)$$

$$3'N(X, Y, Z) = 4[(D_{\bar{X}}{}'F)(Y, Z) + (D_{\bar{Y}}{}'F)(Z, X) + (D_{\bar{Z}}{}'F)(X, Y)]. \quad (2.25b)$$

PROOF: We have in the almost Hermite space

$$'M(X, Y, Z) + 'M(Y, Z, X) + 'M(Z, X, Y) = (D_{\bar{X}}{}'F)(Y, Z) + (D_{\bar{Y}}{}'F)(Z, X) + (D_{\bar{Z}}{}'F)(X, Y) + (D_X{}'F)(Y, \bar{Z}) + (D_Y{}'F)(Z, \bar{X}) + (D_Z{}'F)(X, \bar{Y}). \quad \ldots \quad (2.26)$$

Applying (2.23) in this equation and remembering the fact that $'M$ is completely skew-symmetric in an almost Tachibana space, we have (2.25*a*). (2.25*b*) is obvious from (2.22).

Theorem 2.8—Let us put

$$G(X, Y, Z) \overset{\text{def}}{=} (D_X{}'F)(Y, Z) + (D_Y{}'F)(X, Z). \quad \ldots \quad (2.27)$$

Then

$$'N(Z, X, Y) + 'N(Z, Y, X) = G(X, Y, \bar{Z}) - G(\bar{X}, \bar{Y}, \bar{Z}). \quad \ldots \quad (2.28)$$

PROOF: From (1.18), we have

$$'M(X, Y, Z) + 'M(Y, X, Z) = (D_{\bar{X}}{}'F)(Y, Z) + (D_X{}'F)(Y, \bar{Z}) + (D_{\bar{Y}}{}'F)(X, Z) + (D_Y{}'F)(X, \bar{Z}). \quad \ldots \quad (2.29)$$

Barring X, Y and Z in (2.27) and using (1.20), we get

$$G(\bar{X}, \bar{Y}, \bar{Z}) = -[(D_{\bar{X}}{}'F)(Y, Z)+(D_{\bar{Y}}{}'F)(X, Z)]. \quad .. \quad (2.30)$$

Substituting from (2.30) and (2.27) in (2.29), we get

$$'M(X, Y, Z)+'M(Y, X, Z) = G(X, Y, \bar{Z})-G(\bar{X}, \bar{Y}, \bar{Z}). \quad .. \quad (2.31)$$

We have (Mishra 1967)

$$'M(X, Y, Z)+'M(Y, X, Z) = 'N(Z, X, Y)+'N(Z, Y, X). \quad .. \quad (2.32)$$

(2.28) follows from (2.31) and (2.32).

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