ON THE YIELD CONDITION AND TRANSITION FIELDS IN ELASTIC-PLASTIC DEFORMATION OF SOLIDS*

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Using Seth's concepts of elastic-plastic transition, an invariant relation among the strain (stress) invariants is obtained. Most of the yield conditions of solids existing in the present literature come out as special cases from this relation and, therefore, it may be regarded as a general yield condition applicable to all types of material irrespective of their properties. It has also been shown that transition fields, naturally being non-linear in character, are sub-harmonic (superharmonic) fields.

1. INTRODUCTION

When a continuum changes from one state to another state under external or internal traction, a constraint is placed on the invariants of the strain (stress) tensors such that

$$f(J_1, J_2, J_3) = 0 \quad .. \quad .. \quad .. \quad .. \quad (1.1)$$

where J's are the three invariants of the strain tensor e_{ij}. The constraint (1.1) should hold at transition irrespective of any type of medium, isotropic, anisotropic, homogeneous or heterogeneous, and can be derived from the transition condition independent of the constitutive equations and momentum equations. The strain (stress) invariants, in general, are independent of each other; a functional relation exists between them only at the transition state because of the constraint. Now to express (1.1) in terms of the stress invariants we need a constitutive relation. Let I_1, I_2, I_3 be invariants of the stress tensor τ_{ij}, then (1.1) can be expressed as

$$g(I_1, I_2, I_3) = 0 \quad .. \quad .. \quad .. \quad .. \quad (1.2)$$

with the help of any constitutive equation. The constraint (1.2) holds at transition for any type of isotropic and homogeneous medium and can be derived from transition condition and constitutive relation.

Several *ad hoc* forms for g are widely in use in literature such as that due to Tresca, von Mises, Haar-Kármán among others. Most of the yield conditions of solids existing in literature so far are considered to be self-evident as a

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result of long experimental consequences. Very few authors in this field thus far have attempted to establish them on the basis of analytical consideration.

Following some of the works of Seth (1963, 1966, 1968) in this field, we have shown that the transition region, in fact, is a singular region and the axiom of continuity holds good asymptotically. Therefore, if the plastic region is taken as an image of elastic region under certain transformation, then the Jacobian of transformation should behave singularly. An invariant relation among stress (strain) invariants is obtained, from which most of the yield conditions existing in the literature are deduced as special cases. This is an attempt to establish the yield conditions of solids through analytical consideration. We have also shown that transition fields are sub-harmonic (superharmonic) fields.

2. General Treatment of Elastic-Plastic Transition corresponding to the Vanishing of the Jacobian of Transformation

In recent years, Seth (1963, 1966, 1968) has introduced a new concept of transition in continuum mechanics and has successfully explained that transition phenomenon is asymptotic in nature and, therefore, such problems should be treated as limiting cases. It is well known that when a material yields, it loses all or part of its original intrinsic properties, and hence it is possible that an entirely different material may result. Therefore, the plastic yielding of a material is a consequence of the collapse of its internal macroscopic structure. Geometrically, the transition state corresponds to the degeneracy of the reciprocal strain ellipsoid (Purushothama 1965). The reciprocal strain ellipsoid may reduce to an infinite cylinder, a point or a pair of planes, depending on the particular physical situation.

The transition region, therefore, is a singular region and the axiom of continuity holds good asymptotically. If the plastic region is assumed to be an image of the elastic region under certain transformation, at transition, the Jacobian of transformation should behave singularly. A pre-marked elementary particle in the elastic region cannot be recognized in the plastic region. Therefore, one-to-one correspondence between the elementary particles of the elastic and plastic region fails to exist. As a result, the Jacobian of transformation behave singularly and hence its vanishing or becoming infinite can be regarded as a transition condition corresponding to asymptotically large extension or contraction of the macroscopic element.

If u, v and w are the displacements along the rectangular Cartesian coordinates axes, then

$$X = x-u, \quad Y = y-v, \quad Z = z-w,$$

(X, Y, Z) and (x, y, z) being the coordinates of a point in the undeformed and deformed state. Hence the Jacobian,

$$J = \frac{(x-u, y-v, z-w)}{(x, y, z)} \quad .. \quad .. \quad .. \quad .. \quad (2.1)$$

referred to the deformed state and

$$J = \frac{(X+u, Y+v, z+w)}{(X, Y, Z)} \quad .. \quad .. \quad .. \quad (2.2)$$

referred to the undeformed state.

From (2.1) we get

$$J^2 = \begin{vmatrix} 1-2e_{xx} & -2e_{xy} & -2e_{xz} \\ -2e_{yx} & 1-2e_{yy} & -2e_{yz} \\ -2e_{zx} & -2e_{zy} & 1-2e_{zz} \end{vmatrix}$$

where

$$1-2e_{xx} = -2\frac{\partial u}{\partial x}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2,$$

$$-2e_{xy} = -\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \text{ etc.}$$

The vanishing of (2.1), which is a transition condition corresponding to the asymptotically large extension, gives

$$8J_3-4J_2+2J_1 = 1 \quad .. \quad .. \quad .. \quad .. \quad (2.3)$$

where J_1, J_2, J_3 are three strain invariants given by

$$\left.\begin{aligned} J_1 &= \frac{1}{1!}\delta^i_j e^j_i \\ J_2 &= \frac{1}{2!}\delta^{ik}_{jl} e^l_k e^j_i \\ J_3 &= \frac{1}{3!}\delta^{ikm}_{jln} e^n_m e^l_k e^j_i \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (2.4)$$

e_{ij} are Euclerian strain tensors, δ^i_j, δ^{ik}_{jl}, etc., are generalized Kronecker deltas (Eringen 1962).

A similar condition like (2.3) may be obtained from (2.2) which corresponds to the asymptotically large contraction. If dV and dv are the volume elements in the initial and deformed state respectively, then

$$dV = (1-2J_1+4J_2-8J_3)^{\frac{1}{2}}dv.$$

It seems from the above relation that the undeformed volume element appears to be zero, then if dv is finite thus a material volume element has been created in the deformation process. However, this is not true. As the Jacobian of transformation tends to zero, dv tends to become so large that dv remains finite.

From the geometrical point of view, we again get a condition of the form (2.3). The square of the principal axes of the reciprocal strain ellipsoid are given by

$$1/(1-2e_{ii})$$

where e_{ii} are the principal strains. So, $1-2e_{11} \to 0$ corresponds to infinite extension of one of the axes of the principal strain ellipsoid. Now,

$$J_1 = e_{11}+e_{22}+e_{33}$$
$$J_2 = e_{11}\,e_{22}+e_{22}\,e_{33}+e_{33}\,e_{11}$$
$$J_3 = e_{11}\,e_{22}\,e_{33}\,.$$

Putting $e_{11} = \frac{1}{2}$ in the above relations and eliminating e_{22} and e_{33}, we obtain the same condition as given in (2.3). This clearly suggests that vanishing of the Jacobian of transformation is a transition condition.

The constraint (2.3) holds at transition irrespective of any type of medium, isotropic, anisotropic, homogeneous or heterogeneous, and comes out from the transition condition independent of the constitutive equations and the momentum equations.

To express it in terms of the stress invariants, we need a constitutive equation. Any constitutive equation may be used, but many metals tend to become plastic even while the strain is sufficiently small. Seth (1950) emphasizes that in most materials, the linear law with non-linear components of the strain tensor is quite satisfactory. Therefore, we assume the tensorially linear constitutive equation

$$\tau_{ij} = \lambda\, e_{\alpha\alpha}\,\delta_{ij}+2\mu\, e_{ij}$$

where $e_{\alpha\alpha}$ is the first principal strain invariant and λ, μ are Lame's constants.

If the corresponding stress invariants are I_1, I_2, I_3, then we have

$$J_1 = \frac{1-2\sigma}{E} I_1$$

$$J_2 = \frac{1}{E^2}\left[(1+\sigma)^2 I_2-\sigma(2-\sigma) I_1^2\right]$$

$$J_3 = \frac{1}{E^3}\left[(1+\sigma)^3 I_3-\sigma(1+\sigma)^2 I_1 I_2+\sigma^2 I_1^3\right],$$

E being Young's modulus of elasticity and σ being Poisson's ratio. Substituting these values in (2.3) and putting

$$I_2^1 = \Sigma(\tau_{11}-\tau_{22})^2 = 2 I_1^2-6 I_2,$$

$$I_1/E = K_1, \quad I_2/E^2 = K_2, \quad I_3/E^3 = K_3 \quad \text{and} \quad K_2^1 = I_2^1/E^2,$$

we have

$$8\left[(1+\sigma)^3 K_3+\tfrac{1}{6}\sigma(1+\sigma)^2 K_1 K_2^1-\tfrac{1}{3}\sigma(\sigma^2-\sigma+1) K_1^3\right]$$
$$+\tfrac{4}{3}\left[\tfrac{1}{2}(1+\sigma)^2 K_2^1-(1-2\sigma)^2 K_1^2\right]+2(1-2\sigma) K_1 = 1. \qquad .. \quad (2.5)$$

This invariant relation among the stress invariants contains the elastic effect. For the fully plastic state $\sigma \to \frac{1}{2}$ (condition of incompressibility) we have

$$3K_2^1+2\left(27K_3+\tfrac{3}{2}K_1K_2^1-K_1^3\right)=2. \qquad (2.6)$$

Rewriting (2.6), where $T_{ij}=\tau_{ij}/E$, we get

$$3[(T_{11}-T_{22})^2+(T_{22}-T_{33})^2+(T_{11}-T_{33})^2]$$
$$+2[(2T_{11}-T_{22}-T_{33})(2T_{22}-T_{11}-T_{33})(2T_{33}-T_{11}-T_{22})]=2. \qquad (2.7)$$

Equation (2.7) can alternatively be written as follows:

$$L_1^2+L_2^2+L_3^2+2L_1L_2L_3=2 \qquad (2.8a)$$

where $L_1=2T_{11}-T_{22}-T_{33}$, etc.

Equation (2.7) could be taken as the general yield condition for all types of media irrespective of their intrinsic properties. We shall now consider the following special cases:

I. *Principal Line Theory*

For principal line theory, one of the L's vanishes and (2.8a) reduces to Tresca's type:

$$\tau_{11}-\tau_{33}=\tfrac{2}{3}Y,$$

where Y is yield stress in tension and $E=2Y$.

II. *Haar-Kármán Hypothesis*

If $T_{11}=T_{22}$, from (2.7)

$3(T_{11}-T_{33})^2-2(T_{11}-T_{33})^3=1$ which gives

$T_{11}-T_{33}=1,\ 1,\ -\frac{1}{2}$.

Hence we have the following corresponding to the roots:

(i) $\tau_{33}-\tau_{11}=-2Y, \quad \tau_{11}\geqslant\tau_{22}\geqslant\tau_{33}$

(ii) $\tau_{33}-\tau_{11}=Y, \quad \tau_{11}\leqslant\tau_{22}\leqslant\tau_{33}$.

Both (i) and (ii) are yield conditions of Tresca's type. While Tresca's yield condition does not distinguish between yielding in tension and in compression, we have from our analysis (i) representing yield condition in compression and (ii) representing yield condition in tension.

III. *Plane Strain*

In this case $e_{33}=0$, so $J_{33}=0$ and we get from (2.3)

$4J_2-2J_1+1=0$.

In the limiting incompressible case, $\sigma \to \frac{1}{2}$, we get from (2.6)

$K_2^1=\frac{2}{3}$ which gives

$$(\tau_{11}-\tau_{22})^2+(\tau_{22}-\tau_{33})^2+(\tau_{33}-\tau_{11})^2=\frac{8Y^2}{3} \qquad (2.8)$$

which is exactly of the same form as von Mises yield condition:

$$(\tau_{11}-\tau_{22})^2+(\tau_{22}-\tau_{33})^2+(\tau_{33}-\tau_{11})^2 = 2Y^2. \quad .. \quad .. \quad (2.9)$$

We notice here that the von Mises yield condition is applicable only in the case of plane strain and that the constant $2Y^2$ in (2.9) should be replaced by $\frac{8Y^2}{3}$.

IV. *Longitudinal Strain*

If $T_{22} = 0$ and $T_{33} = 0$, it follows from (2.7)

$3T_{11}^2+2T_{11}^3 = 1$ and the roots are $-1, -1, \frac{1}{2}$, hence

$\tau_{11} = -2Y$, $\tau_{11} = Y$, which are yield stresses in compression and tension respectively.

V. *Plane Stress*

In this case, suppose $T_{33} = 0$, and the yield condition (2.6) will reduce to $(T_{11}+T_{22}+1)(2T_{11}-T_{22}-1)(2T_{22}-T_{11}-1) = 0$. We again get the condition of the Tresca type.

3. Sub-harmonicity of Transition Fields

Any transition in harmonic or bi-harmonic fields, which permeate natural phenomena, may be expected to exhibit itself in terms of the allied sub-harmonic or superharmonic fields. The latter are non-linear and non-conservative in nature. Since we are interested in this work only in elastic-plastic transition, this section will be devoted to showing that the elastic-plastic transition field is a sub-harmonic (superharmonic) field.

For simplicity, we are using here a rectangular Cartesian frame of reference. It may be extended to general curvilinaear coordinate system. The deformation field is given by

$$u^r = x^r - X^r, \quad (r = 1, 2, 3) \quad .. \quad .. \quad .. \quad (3.1)$$

where x^r, X^r are respectively the deformed and undeformed coordinates of a point and u^r are displacement of the point along the coordinate axes. If e_{ij} is the strain tensor, then using Almansi strain measure, we have

$$\begin{aligned} 2e_{ij} &= u_{i;j}+u_{j;i}-u_{\alpha;j}u_{\alpha;i} \\ &= \delta_{ij}-X^{\alpha}_{,i}X^{\alpha}_{,j}. \end{aligned}$$

The covariant differentiation reduces to ordinary partial differentiation because of the choice of rectangular Cartesian coordinates axes. The stress-strain relation $\tau_{ij} = \lambda\, e_{\alpha\alpha}\, \delta_{ij}+2\mu\, e_{ij}$ becomes

$$\tau_{ij} = \lambda\, e_{\alpha\alpha}\, \delta_{ij}+\mu\left[\delta_{ij}-X^{\alpha}_{,i}X^{\alpha}_{,j}\right] \quad .. \quad .. \quad .. \quad (3.2)$$

where

$$2e_{\alpha\alpha} = 3-X^{\alpha}_{,i}\,X^{\alpha}_{,i}; \quad i, j = 1, 2, 3. \quad .. \quad .. \quad (3.2a)$$

Now, in the absence of body forces and if only a steady state deformation is assumed, the equilibrium equation may be written as

$$\tau_{ij,\,j} = 0. \qquad .. \quad .. \quad .. \quad .. \quad (3.3)$$

Hence using eqns. (3.2) in (3.3), we obtain the following:

$$\begin{aligned}\frac{(\lambda+2\mu)}{\mu}\frac{\partial}{\partial x}(e_{\alpha\alpha}) = {} & J_{xy}(X^1;\, X^1_{,\,y})+J_{xz}(X^1;\, X^1_{,\,z}) \\ & +J_{xy}(X^2;\, X^2_{,\,y})+J_{xz}(X^2;\, X^2_{,\,z}) \\ & +J_{xy}(X^3;\, X^3_{,\,y})+J_{xz}(X^3;\, X^3_{,\,z}) \qquad .. \quad (3.4)\end{aligned}$$

$$\begin{aligned}\frac{(\lambda+2\mu)}{\mu}\frac{\partial}{\partial y}(e_{\alpha\alpha}) = {} & J_{yx}(X^1;\, X^1_{,\,x})+J_{yz}(X^1;\, X^1_{z}) \\ & +J_{yx}(X^2;\, X^2_{,\,x})+J_{yz}(X^2;\, X^2_{,\,z}) \\ & +J_{yx}(X^3;\, X^3_{,\,x})+J_{yz}(X^3;\, X^3_{z}) \qquad .. \quad (3.5)\end{aligned}$$

and

$$\begin{aligned}\frac{(\lambda+2\mu)}{\mu}\frac{\partial}{\partial z}(e_{\alpha\alpha}) = {} & J_{zx}(X^1;\, X^1_{,\,x})+J_{zy}(X^1;\, X^1_{,\,y}) \\ & +J_{zx}(X^2;\, X^2_{,\,x})+J_{zy}(X^2;\, X^2_{,\,y}) \\ & +J_{zx}(X^3;\, X^3_{,\,x})+J_{zy}(X^3;\, X^3_{,\,y}). \qquad .. \quad (3.6)\end{aligned}$$

The above three equations may be written as

$$\begin{aligned} & e_{\alpha\alpha,\,i} = \tfrac{1}{2}c\, J_{ij}(X^r;\, X^r_{,\,j}), \qquad .. \quad .. \quad .. \quad (3.7) \\ & J_{ii} = 0 \text{ (no sum on } i), \end{aligned}$$

where

$$c = \frac{2\mu}{\lambda+2\mu} = \frac{1-2\sigma}{1-\sigma}; \quad r, i, j = 1, 2, 3,$$

and

$$J_{xy}(X^1;\, X^1_{,\,y}) = \begin{vmatrix} \dfrac{\partial X^1}{\partial x} & \dfrac{\partial^2 X^1}{\partial x\,\partial y} \\ \dfrac{\partial X^1}{\partial y} & \dfrac{\partial^2 X^1}{\partial y^2} \end{vmatrix}, \text{ etc.}$$

Now differentiating (3.4) with respect to y and (3.5) with respect to x and then subtracting, we get

$$J_{xy}[X^1;\, \nabla^2 X^1]+J_{xy}[X^2;\, \nabla^2 X^2]+J_{xy}[X^3;\, \nabla^2 X^3] = 0. \qquad .. \quad (3.8)$$

A similar treatment will give two more equations like (3.8). The three equations just mentioned may be written as

$$\begin{aligned} & J_{ij}[X^r;\, \nabla^2 X^r] = 0 \qquad .. \quad .. \quad .. \quad .. \quad .. \quad (3.9) \\ & J_{ii} = 0 \text{ (no sum on } i), \quad r, i, j = 1, 2, 3. \end{aligned}$$

The general solution T of (3.9), when it exists, may be written in the form

$$\nabla^2(X^r) = \nabla T(X^r) \quad .. \quad .. \quad .. \quad .. \quad (3.10)$$

where

$$\nabla = i\frac{\partial}{\partial X^1}+j\frac{\partial}{\partial X^2}+K\frac{\partial}{\partial X^3}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

provided $J_{ij}(X^r, X^s)$, $i \neq j$, $r \neq s$, is finite.
The equation (3.10) shows the sub-harmonicity (superharmonicity) of the transition field.

Equation (3.7) can be integrated with the help of eqns. (3.10) and thus we obtain the dilatation,

$$e_{\alpha\alpha} = \tfrac{1}{2}c\left[T-\tfrac{1}{2}(X^r{}_{,i}\, X^r{}_{,i})\right], \quad i, r = 1, 2, 3.$$

Using (3.2a) we have

$$e_{\alpha\alpha} = \frac{c}{2-c}\left[T-\tfrac{3}{2}\right].$$

For fully plastic state $c \to 0$ and hence $e_{\alpha\alpha} \to 0$, which is a condition of incompressibility.

4. Discussion of the Results and Conclusion

The classical yield conditions remain the same in tension and in compression, while it is known that the yield stress in compression may not be equal in tension—in all situations. Most of the yield conditions are obtained on the basis of experimentation. They are *ad hoc* relation and bear no relation to the geometry of the field. The results in this paper are obtained from the analytic consideration and make distinction between yield stress in tension and compression.

Equation (2.4), which is obtained from one of the transition conditions, is an invariant relation and all of the results are derived from it. Therefore, it seems reasonable to say that in the Huber-von Mises-Hencky yield condition the constant should be $\frac{8}{3}Y^2$ rather than the classical value $2Y^2$.

Also it is seen that the aforesaid yield condition holds only in the case of plane strain and Tresca yield condition is a special case of the von Mises yield condition when two of the principal stresses are equal.

When a material starts to yield, the elastic effect may still be present and only in the ideal plastic material is it assumed to be completely absent. The classical yield conditions were not derived from a transition condition in which elastic effects may be present and the medium is not completely incompressible. Our yield condition (2.5) does contain elastic effect. However, we like to emphasize that in transition treatment of elastic-plastic problems no yield conditions are utilized. The above discussion shows that the yield condition, when it exists, follows from the fundamental equations.

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