ON ALMOST COMPLEX SPACES

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In this paper we have obtained an almost F-connection D and deduced its properties.

We consider a $2n$-dimensional space M_{2n} of differentiability class C^{r+1} and have the following agreement.

Agreement (1.1)—All the equations which follow hold for arbitrary vector fields X, Y, Z, . . . , etc.

Let there be defined in M_{2n} a vector-valued linear function F, such that if

$$\overline{X} \overset{\text{def}}{=} F(X) \qquad (1)$$

then

$$\overline{\overline{X}}+X = 0. \qquad (2)$$

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an almost complex space.

In an almost complex space a connection D is said to be an F-connection (Yano 1965) if

$$(D_X F)(Y) = 0. \qquad (3)$$

In view of (1) and (2), this equation is equivalent to

$$D_X \overline{Y} = \overline{D_X Y}, \quad D_{\overline{X}} \overline{Y} = \overline{D_{\overline{X}} Y}, \quad -D_{\overline{X}} Y = \overline{D_{\overline{X}} \overline{Y}}, \quad -D_X Y = \overline{D_X \overline{Y}}.$$

In an almost complex space M_{2n}, let there be defined a Hermite metric g:

$$g(\overline{X}, \overline{Y}) = g(X, Y). \qquad (4a)$$

Then M_{2n} is said to be an almost Hermite space. By virtue of (2) eqn. (4*a*) is equivalent to

$$'F(X, Y) \overset{\text{def}}{=} g(\overline{X}, Y) = -g(X, \overline{Y}) \qquad (4b)$$

$$'F(X, \overline{Y}) = -'F(\overline{X}, Y) = g(X, Y) \qquad (4c)$$

$$'F(\overline{X}, \overline{Y}) = 'F(X, Y). \qquad (4d)$$

From (4*b*) we immediately prove that $'F$ is skew-symmetric:

$$'F(X, Y)+'F(Y, X) = 0.$$

In an almost Hermite space a connection D is called an almost F-connection (Mishra 1969) if

$$(D_X{'F})(Y, Z)+(D_Y{'F})(Z, X)+(D_Z{'F})(X, Y) = 0. \quad \text{(5)}$$

In consequence of (1), (2) and (4), eqn. (5) is equivalent to

$$g(D_X\bar{Y}, Z)+g(D_Y\bar{Z}, X)+g(D_Z\bar{X}, Y) = {'F}(D_XY, Z)+{'F}(D_YZ, X)+{'F}(D_ZX, Y)$$
$$g(D_{\bar{X}}\bar{Y}, Z)-{'F}(D_Y\bar{Z}, X)-g(D_ZX, Y) = {'F}(D_{\bar{X}}Y, Z)+g(D_YZ, X)+{'F}(D_Z\bar{X}, Y)$$
$$-g(D_{\bar{X}}Y, Z)-{'F}(D_{\bar{Y}}\bar{Z}, X)+{'F}(D_ZX, Y) = {'F}(D_{\bar{X}}\bar{Y}, Z)+g(D_{\bar{Y}}Z, X)+g(D_Z\bar{X}, Y)$$
$${'F}(D_{\bar{X}}Y, Z)+{'F}(D_{\bar{Y}}Z, X)+{'F}(D_{\bar{Z}}X, Y) = g(D_{\bar{X}}\bar{Y}, Z)+g(D_{\bar{Y}}\bar{Z}, X)+g(D_{\bar{Z}}\bar{X},Y).$$

Let S be the torsion tensor of D:

$$S(X, Y) = D_XY-D_YX-[X, Y]. \quad \text{(6)}$$

Then D is called half-symmetric if

$$S(X, Y)-S(\bar{X}, \bar{Y}) = \overline{S(X, \bar{Y})}+\overline{S(\bar{X}, Y)} \quad \text{(7a)}$$

or

$$S(X, \bar{Y})+S(\bar{X}, Y) = \overline{S(\bar{X}, \bar{Y})}-\overline{S(X, Y)}. \quad \text{(7b)}$$

Let us put (Mishra 1969)

$$U(X) \overset{\text{def}}{=} (C_1^1\bar{S})(X). \quad \text{(8a)}$$

Then if

$$nS(X, Y) = XU(Y)-YU(X)+\bar{X}U(\bar{Y})-\bar{Y}U(\bar{X}) \quad \text{(8b)}$$

D is called almost symmetric.

Let us put

$$T(X) \overset{\text{def}}{=} (C_1^1S)(X). \quad \text{(9a)}$$

Then if

$$nS(X, Y) = XT(Y)-YT(X)+\bar{X}T(\bar{Y})-\bar{Y}T(\bar{X}) \quad \text{(9b)}$$

D is called semi-symmetric.

We have (Mishra 1969)

$$\begin{aligned} S(X, Y)-S(\bar{X}, \bar{Y})-\overline{S(X, \bar{Y})}-\overline{S(\bar{X}, Y)} &= \{[X, Y]-[\bar{X}, \bar{Y}]-\overline{[X, \bar{Y}]}-\overline{[\bar{X}, Y]}\} \\ &\times(\alpha-\delta-1)+\{\overline{[X, Y]}-\overline{[\bar{X}, \bar{Y}]}+[X, \bar{Y}]+[\bar{X}, Y]\}(\theta-\sigma)+\{s(X, Y) \\ &-s(\bar{X}, \bar{Y})-\overline{s(X, \bar{Y})}-\overline{s(\bar{X}, Y)}(\alpha-\delta-\phi-\rho)+\{\overline{s(X, Y)}-\overline{s(\bar{X}, \bar{Y})} \\ &+s(X, \bar{Y})+s(\bar{X}, Y)\}(\theta-\sigma+\beta+\gamma). \end{aligned} \quad \text{(10)}$$

Nijenhuis tensor N of the connection E in the almost complex space M_{2n} is given by

$$N(X, Y) = E_{\bar{X}}\bar{Y}-E_{\bar{Y}}\bar{X}-E_XY+E_YX-\overline{E_X\bar{Y}}+\overline{E_{\bar{Y}}X}-\overline{E_{\bar{X}}Y}+\overline{E_Y\bar{X}}. \quad \text{(11)}$$

It can be verified that

$$N(X, Y) = \overline{N(X, \bar{Y})} = -N(\bar{X}, \bar{Y}) = \overline{N(\bar{X}, Y)}. \quad \text{(12)}$$

Theorem (1.1)—Let connections D and E be related by

$$D_X Y \overset{\text{def}}{=} \alpha E_X Y + \beta E_X \bar{Y} + \gamma E_{\bar{X}} Y + \delta E_{\bar{X}} \bar{Y} + \theta \overline{E_X Y} + \phi \overline{E_X \bar{Y}} + \rho \overline{E_{\bar{X}} Y} + \sigma \overline{E_{\bar{X}} \bar{Y}}. \quad \ldots \quad (13)$$

If any two of the following properties hold, the third also holds:

(i) connection D is almost symmetric

(ii) connection E is almost symmetric

(iii) $[\bar{X}, \bar{Y}] - [X, Y] + \overline{[X, \bar{Y}]} + \overline{[\bar{X}, Y]} = 0$.

Proof: The statement follows from (8) and (10).

Theorem (1.2)—Let E be an arbitrary connection. Then the connection D given by

$$'F(D_X Y, Z) = \alpha[{'F}(E_X Y, Z) + g(E_X \bar{Y}, Z)] + \beta[{'F}(E_X \bar{Y}, Z) - g(E_X Y, Z)]$$
$$+ \gamma[{'F}(E_{\bar{X}} Y, Z) + g(E_{\bar{X}} \bar{Y}, Z)] + \delta[{'F}(E_{\bar{X}} \bar{Y}, Z) - g(E_{\bar{X}} Y, Z)] \quad \ldots \quad (14)$$

is an almost F-connection.

Proof: Let us put

$$D_X Y \overset{\text{def}}{=} \alpha E_X Y + \beta E_X \bar{Y} + \gamma E_{\bar{X}} Y + \delta E_{\bar{X}} \bar{Y} + \theta \overline{E_X Y} + \phi \overline{E_X \bar{Y}} + \rho \overline{E_{\bar{X}} Y} + \sigma \overline{E_{\bar{X}} \bar{Y}}. \quad (15)$$

Barring y in this equation, we get

$$(D_X F)(Y) + \overline{D_X Y} = \alpha E_X \bar{Y} - \beta E_X Y + \gamma E_{\bar{X}} \bar{Y} - \delta E_{\bar{X}} Y + \overline{\theta E_X \bar{Y}} - \overline{\phi E_X Y} + \overline{\rho E_{\bar{X}} \bar{Y}} - \overline{E_{\bar{X}} Y}. \quad \ldots \quad (16)$$

From (4), (16) and (5), we have

$$'F(D_X Y, Z) + {'F}(D_Y Z, X) + {'F}(D_Z X, Y) = \alpha[g(E_X \bar{Y}, Z) + g(E_Y \bar{Z}, X) + g(E_Z \bar{X}, Y)]$$
$$- \beta[g(E_X Y, Z) + g(E_Y Z, X) + g(E_Z X, Y)] + \gamma[g(E_{\bar{X}} \bar{Y}, Z) + g(E_{\bar{Y}} \bar{Z}, X)$$
$$+ g(E_{\bar{Z}} \bar{X}, Y)] - \delta[g(E_{\bar{X}} Y, Z) + g(E_{\bar{Y}} Z, X) + g(E_{\bar{Z}} X, Y)] + \theta[{'F}(E_X \bar{Y}, Z)$$
$$+ {'F}(E_Y \bar{Z}, X) + {'F}(E_Z \bar{X}, Y)] - \phi[{'F}(E_X Y, Z) + {'F}(E_Y Z, X) + {'F}(E_Z X, Y)]$$
$$+ \rho[{'F}(E_{\bar{X}} \bar{Y}, Z) + {'F}(E_{\bar{Y}} \bar{Z}, X) + {'F}(E_{\bar{Z}} \bar{X}, Y)] - \sigma[{'F}(E_{\bar{X}} Y, Z) + {'F}(E_{\bar{Y}} Z, X)$$
$$+ {'F}(E_{\bar{Z}} X, Y)]. \quad \ldots \quad (17)$$

From (4) and (15), we have

$$'F(D_X Y, Z) + {'F}(D_Y Z, X) + {'F}(D_Z X, Y) = \alpha[{'F}(E_X Y, Z) + {'F}(E_Y Z, X)$$
$$+ {'F}(E_Z X, Y)] + \beta[{'F}(E_X \bar{Y}, Z) + {'F}(E_Y \bar{Z}, X) + {'F}(E_Z \bar{X}, Y)]$$
$$+ \gamma[{'F}(E_{\bar{X}} Y, Z) + {'F}(E_{\bar{Y}} Z, X) + {'F}(E_{\bar{Z}} X, Y)] + \delta[{'F}(E_{\bar{X}} \bar{Y}, Z)$$
$$+ {'F}(E_{\bar{Y}} \bar{Z}, X) + {'F}(E_{\bar{Z}} \bar{X}, Y)] - \theta[g(E_X Y, Z) + g(E_Y Z, X)$$
$$+ g(E_Z X, Y)] - \phi[g(E_X \bar{Y}, Z) + g(E_Y \bar{Z}, X) + g(E_Z \bar{X}, Y)]$$
$$- \rho[g(E_{\bar{X}} Y, Z) + g(E_{\bar{Y}} Z, X) + g(E_{\bar{Z}} X, Y)] - \sigma[g(E_{\bar{X}} \bar{Y}, Z)$$
$$+ g(E_{\bar{Y}} \bar{Z}, X) + g(E_{\bar{Z}} \bar{X}, Y)]. \quad \ldots \quad (18)$$

Comparing the coefficients, we get

$$\alpha = -\phi, \quad \theta = \beta, \quad \sigma = -\gamma, \quad \rho = \delta.$$

Substituting from the equations in (15) and using (4), we get (14).

Corollary (1.1)—Let two connections D and E be related by (15). The necessary and sufficient condition that one connection is almost F-connection is that the other is also almost F-connection provided

$$(\alpha+\phi)(\theta-\beta)(\gamma+\sigma)(\rho-\delta) \neq 0.$$

Proof: We have

$$\begin{aligned}(D_X{}'F)(Y, Z)+(D_Y{}'F)(Z, X)+(D_Z{}'F)(X, Y) = {} & (\alpha+\phi)[(E_X{}'F)(Y, Z) \\ & +(E_Y{}'F)(Z, X)+(E_Z{}'F)(X, Y)]+(\theta-\beta)[(E_X{}'F)(Y, \bar{Z}) \\ & +(E_Y{}'F)(Z, \bar{X})+(E_Z{}'F)(X, Y)]+(\gamma+\sigma)[(E_{\bar{X}}{}'F)(Y, Z) \\ & +(E_{\bar{Y}}{}'F)(Z, X)+(E_{\bar{Z}}{}'F)(X, Y)]+(\rho-\delta)[(E_{\bar{X}}{}'F)(Y, \bar{Z}) \\ & +(E_{\bar{Y}}{}'F)(Z, \bar{X})+(E_{\bar{Z}}{}'F)(X, \bar{Y})]. \qquad (19)\end{aligned}$$

The statement follows from (5) and (19).

Corollary (1.2)—Equation (14) is equivalent to

$$\begin{aligned}{}'F(D_X Y, \bar{Z}) = {} & -\alpha[{}'F(\overline{E_X Y}, Z)+g(\overline{E_X \bar{Y}}, Z)]+\beta[-{}'F(\overline{E_X \bar{Y}}, Z)+g(\overline{E_X Y}, Z)] \\ & -\gamma[{}'F(\overline{E_{\bar{X}} Y}, Z)+g(\overline{E_{\bar{X}} \bar{Y}}, Z)]+\delta[g(\overline{E_{\bar{X}} Y}, Z)-{}'F(\overline{E_{\bar{X}} \bar{Y}}, Z)] \qquad (20)\end{aligned}$$

$$\begin{aligned}{}'F(D_{\bar{X}} Y, Z) = {} & \alpha[{}'F(E_{\bar{X}} Y, Z)+g(E_{\bar{X}} \bar{Y}, Z)]+\beta[{}'F(E_{\bar{X}} \bar{Y}, Z)-g(E_{\bar{X}} Y, Z)] \\ & -\gamma[{}'F(E_X Y, Z)+g(E_X \bar{Y}, Z)]+\delta[g(E_X Y, Z)-{}'F(E_X \bar{Y}, Z)] \qquad (21)\end{aligned}$$

$$\begin{aligned}{}'F(D_X \bar{Y}, Z) = {} & \alpha[{}'F(E_X \bar{Y}, Z)-g(E_X Y, Z)]-\beta[{}'F(E_X Y, Z)-g(E_X \bar{Y}, Z)] \\ & +\gamma[{}'F(E_{\bar{X}} \bar{Y}, Z)-g(E_{\bar{X}} Y, Z)]-\delta[{}'F(E_{\bar{X}} Y, Z)+g(E_{\bar{X}} \bar{Y}, Z)] \qquad (22)\end{aligned}$$

$$\begin{aligned}{}'F(D_{\bar{X}} \bar{Y}, Z) = {} & \alpha[{}'F(E_{\bar{X}} \bar{Y}, Z)-g(E_{\bar{X}} Y, Z)]-\beta[{}'F(E_{\bar{X}} Y, Z)+g(E_{\bar{X}} \bar{Y}, Z)] \\ & +\gamma[g(E_X Y, Z)-{}'F(E_X \bar{Y}, Z)]+\delta[{}'F(E_X Y, Z)+g(E_X \bar{Y}, Z)] \qquad (23)\end{aligned}$$

$$\begin{aligned}{}'F(D_{\bar{X}} \bar{Y}, \bar{Z}) = {} & \alpha[{}'F(E_{\bar{X}} \bar{Y}, \bar{Z})-g(E_{\bar{X}} Y, \bar{Z})]-\beta[{}'F(E_{\bar{X}} Y, \bar{Z})+g(E_{\bar{X}} \bar{Y}, \bar{Z})] \\ & +\gamma[g(E_X Y, \bar{Z})-{}'F(E_X \bar{Y}, \bar{Z})]+\delta[{}'F(E_X Y, \bar{Z})+g(E_X \bar{Y}, \bar{Z})]. \qquad (24)\end{aligned}$$

Proof: Barring different vectors in (14) and using (2), we obtain (20)–(24).

Theorem (1.3). For almost symmetric connection D, we have

$$\begin{aligned}n[S(X, Y)+S(\bar{X}, Y)] = {} & U(Y)(X+\bar{X})+U(\bar{Y})(\bar{X}-X)-Y(U(X)+U(\bar{X})) \\ & -\bar{Y}(U(\bar{X})-U(X)). \qquad (25)\end{aligned}$$

Proof: The statement follows from (1), (2) and (8).

Theorem (1.4)—Let S be the torsion tensor of D and s be the torsion tensor of E. Let

$$[X, Y] \overset{\text{def}}{=} E_X Y - E_Y X. \qquad (26)$$

Then

$$S(X, Y)-S(\bar{X}, \bar{Y})-\overline{S(\bar{X}, \bar{Y})}-\overline{S(\bar{X}, Y)} = (N(X, Y)+2[X, Y]-2[\bar{X}, \bar{Y}])$$
$$\times(\alpha-\delta-1)+(\overline{N(X, Y)}+2\overline{[X, Y]}-2\overline{[\bar{X}, \bar{Y}]})(\theta-\sigma)+(s(X, Y)-s(\bar{X}, \bar{Y})$$
$$-\overline{s(X, \bar{Y})}-\overline{s(\bar{X}, Y)})(\alpha-\delta-\phi-\rho)+(\overline{s(X, Y)}-\overline{s(\bar{X}, \bar{Y})}$$
$$+s(X, \bar{Y})+s(\bar{X}, Y))(\theta-\sigma+\beta+\gamma) \quad \ldots \quad (27a)$$

$$\overline{S(X, Y)}-\overline{S(\bar{X}, \bar{Y})}+S(X, \bar{Y})+S(\bar{X}, Y) = (\overline{N(X, Y)}+2\overline{[X, Y]}-2\overline{[\bar{X}, \bar{Y}]})$$
$$\times(\alpha-\delta-1)+(\overline{s(X, Y)}-\overline{s(\bar{X}, \bar{Y})}+s(X, \bar{Y})+s(\bar{X}, Y))(\alpha-\delta-\phi-\rho)$$
$$+(\overline{s(X, \bar{Y})}+\overline{s(\bar{X}, Y)}-s(X, Y)+s(\bar{X}, \bar{Y})(\theta-\sigma+\beta+\gamma)$$
$$-(N(X, Y)+2[X, Y]-2[\bar{X}, \bar{Y}])(\theta-\sigma). \quad \ldots \quad (27b)$$

Proof: These equations follow immediately from (2), (10) and (11).

Corollary (1.3)—Let us put

$$N(X, Y) = 2[\bar{X}, \bar{Y}]-2[X, Y]. \quad \ldots \quad (28)$$

Then the above eqns. (27) assume the forms

$$S(X, Y)-S(\bar{X}, \bar{Y})-\overline{S(X, \bar{Y})}-\overline{S(\bar{X}, Y)} = [s(X, Y)-s(\bar{X}, \bar{Y})-\overline{s(X, \bar{Y})}$$
$$-\overline{s(\bar{X}, Y)}(\alpha-\delta-\phi-\rho)]+[\overline{s(X, Y)}-\overline{s(\bar{X}, \bar{Y})}+s(X, \bar{Y})$$
$$+s(\bar{X}, Y)(\theta-\sigma+\beta+\gamma)] \quad \ldots \quad (29a)$$

$$\overline{S(X, Y)}-\overline{S(\bar{X}, \bar{Y})}+S(X, \bar{Y})+S(\bar{X}, Y) = [\overline{s(X, Y)}-\overline{s(\bar{X}, \bar{Y})}+s(X, \bar{Y})$$
$$+s(\bar{X}, Y)(\alpha-\delta-\phi-\rho)]+[\overline{s(X, \bar{Y})}+\overline{s(\bar{X}, Y)}-s(X, Y)$$
$$+s(\bar{X}, \bar{Y})(\theta-\sigma+\beta+\gamma)]. \quad \ldots \quad (29b)$$

Proof: Substituting from (28) in (27), we obtain (29).

Corollary (1.4)—When (28) is satisfied, the necessary and sufficient condition that E is half-symmetric is that D is half-symmetric.

Proof: The statement follows from (7) and (29).

Corollary (1.5)—When (28) is satisfied, the necessary and sufficient condition that E is almost symmetric is that D is almost symmetric.

Proof: The statement follows from (8) and (29).

Corollary (1.6)—When (28) is satisfied, the necessary and sufficient condition that E is semi-symmetric is that D is semi-symmetric.

Proof: The statement follows from (9) and (29).

References

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