ON THE SUMMABILITY AND ABSOLUTE SUMMABILITY OF A GENERALIZED LAPLACE-STIELTJES INTEGRAL

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In this paper two theorems concerning the summability and absolute summability of the integral

$$\int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\, dA(t),$$

where $W_{k,m}$ is the Whittaker's function, $A(t)$ is a function of bounded variation in every finite interval $0 \leq t \leq T$ and $s = \sigma + i\tau$ is a complex variable, have been established.

§ 1. Varma (1951) gave a generalization

$$f(s) = \int_0^\infty (st)^{m-1/2} e^{-st/2} W_{k,m}(st)\, dA(t) \quad .. \quad .. \quad (1.1)$$

where $W_{k,m}$ is Whittaker's function (Whittaker and Watson 1946) and $A(t)$ is a function of bounded variation in any finite interval $0 \leq t \leq T$ of the Laplace-Stieltjes integral

$$f(s) = \int_0^\infty e^{-st}\, dA(t). \quad .. \quad .. \quad .. \quad .. \quad (1.2)$$

When $k+m = 1/2$, (1.1) reduces to (1.2). It is also well known that if

$$A(x) = \sum_{\lambda_n \leq x} a_n$$

where $0 < \lambda_0 < \lambda_1 < \ldots$ and $\lambda_n \to \infty$, (1.2) reduces to the Dirichlet's series

$$f(s) = \sum a_n e^{-\lambda_n s}. \quad .. \quad .. \quad .. \quad .. \quad (1.3)$$

Summability of (1.2) and (1.3) were investigated by Riesz (1922–23) and their absolute summability by Obrechkoff (1929). Chandrasekharan and Minakshisundaram (1952) gave a relation between the abscissa of summability (C, α) and abscissa of absolute summability $|C, \alpha+1|$ for (1.3). Corresponding result for (1.2) was given by Bosanquet (1953). The purpose of the present paper is to study (1.1) for these properties.

§ 2. *Notations:* We introduce the following notations:

$$\phi_{k,m}(st) = (st)^{m-1/2}\, e^{-(1/2)st}\, W_{k,m}(st)$$

$$C(x;\, s) = \int_0^x \phi_{k,m}(st)\, dA(t)$$

$$C_\alpha(x;\, s) = \{1/\Gamma(\alpha+1)\} \int_0^x (x-t)^\alpha\, \phi_{k,m}(st)\, dA(t), \qquad (\alpha > 0)$$

$$C_0(x;\, s) = C(x;\, s)$$

$$C_\alpha(x;\, s) = (d/dx)\, C_{\alpha+1}(x;\, s), \qquad (-1 < \alpha < 0)$$

$$\bar{A}(x) = \int_0^x t\, dA(t)$$

$$\bar{A}_\alpha(x) = \{1/\Gamma(\alpha+1)\} \int_0^\infty (x-t)^\alpha\, t\, dA(t), \qquad (\alpha \geq 1)$$

$$= -\{1/\Gamma(\alpha+1)\} \int_0^x A(t)(d/dt)\{(x-t)^\alpha\, t\}\, dt, \quad (\alpha > 1).$$

Thus, if $\alpha \geq 1$, we have

$$\bar{A}_\alpha(x) = \{1/\Gamma(\alpha+1)\} \int_0^x (x-t)^\alpha \{x-(x-t)\}\, dA(t)$$

$$= x\, A_\alpha(x) - (\alpha+1)\, A_{\alpha+1}(x). \quad .. \quad .. \quad .. \quad .. \quad (2.1)$$

If $\alpha > 1$, we denote by h the greatest integer less than α; if $0 < \alpha < 1$, we define $h = 0$ and $h = -1$ if $\alpha = 0$. $[\alpha]$ will denote the integral part of α. $c_0, c_1, c_2, \ldots$ stand for numerical constants, not necessarily the same at each place of occurrence. We shall take k and m to be real and $s = \sigma + i\tau$.

§ 3. *Theorem* 1—If

$$\int_0^\infty \phi_{k,m}(s_0 t)\, dA(t) \quad .. \quad .. \quad .. \quad .. \quad (3.1)$$

is summable (C, α) [or $|C, \alpha|$], then (1.1) is also summable (C, α) [or $|C, \alpha|$], provided $m-(\alpha+1)/2 > 0$, $\alpha > 0$ and $\sigma > \sigma_0 \geq 0$.

Theorem 2—If the integral (1.1) is summable (C, α), $\alpha \geq 0$ for $s = s_0$, then it is summable $|C, \alpha+1|$ for every s such that $\sigma > \sigma_0$.

We shall need the following lemmas:

Lemma 3.1 (Bosanquet 1953)—If $k \geq 0$ and $K(x)$ and $D^k K(x)$ are absolutely continuous, in every finite interval $0 \leq x \leq X$, where

$$D^k K(x) = \frac{(-1)^{[k]+1}}{\Gamma(1+[k]-k)} (d/dx)^{[k]} \lim_{w \to \infty} (d/dx) \int_x^w (u-x)^{[k]-k}\, K(u)\, du$$

the necessary and sufficient conditions for the integral

$$\int_0^\infty K(u)\, dA(u)$$

to be summable (C, k) [or $|C, k|$], whenever $\int_0^\infty dA(u)$ is summable (C, k) [or $|C, k|$], are that $K(x)$ and $x^k D^k K(x)$ should be of bounded variation [or bounded] in $[A, \infty]$, whenever $A > 0$.

Alternative conditions are that

(i) $K(x) = o(1)$ and (ii) $\int_A^\infty x^k D^{k+1} K(x)\,dx$ absolutely convergent [or bounded $(C, 0)$].

Lemma 3.2—If $\alpha \geq 0$, then $D^\alpha \phi_{k,m}(sx)$ is absolutely continuous in $(0, \infty)$, provided $m-(\alpha+1)/2 > 0$.

PROOF:

$$
\begin{aligned}
D^\alpha \phi_{k,m}(x) &= \frac{(-1)^{[\alpha]+1}}{\Gamma(1+[\alpha]-\alpha)} (d/dx)^{[\alpha]} \lim_{w\to\infty} (d/dx) \int_x^w (w-x)^{[\alpha]-\alpha} \phi_{k,m}(su)\,du \\
&= \frac{(-1)^{[\alpha]+1}}{\Gamma(1+[\alpha]-\alpha)} (d/dx)^{[\alpha]} \lim_{w\to\infty} (d/dx) \Bigg[\frac{(w-x)^{[\alpha]-\alpha+1}}{[\alpha]-\alpha+1} \phi_{k,m}(sw) \\
&\quad + \{s/([\alpha]-\alpha+1)\} \int_x^w (u-x)^{[\alpha]-\alpha+1} \phi_{k+1/2,\, m-1/2}(su)\,du \Bigg] \\
&= \frac{(-1)^{[\alpha]+1}}{\Gamma([\alpha]-\alpha+1)} (d/dx)^{[\alpha]} \lim_{w\to\infty} \Bigg[-(w-x)^{[\alpha]-\alpha} \phi_{k,m}(sw) \\
&\quad - s \int_x^w (u-x)^{[\alpha]-\alpha} \phi_{k+1/2,\, m-1/2}(su)\,du \Bigg] \\
&= \frac{(-1)^{[\alpha]}s}{\Gamma(1+[\alpha]-\alpha)} (d/dx)^{[\alpha]} \int_x^\infty (u-x)^{[\alpha]-\alpha} \phi_{k+1/2,\, m-1/2}(su)\,du \\
&= \frac{(-1)^{[\alpha]}s}{\Gamma(1+[\alpha]-\alpha)} (d/dx)^{[\alpha]} \int_0^\infty v^{[\alpha]-\alpha} \phi_{k+1/2,\, m-1/2}\, s(v+x)\,dv \\
&= \frac{(s)^{[\alpha]+1}}{\Gamma(1+[\alpha]-\alpha)} \int_0^\infty v^{[\alpha]-\alpha} \phi_{k+([\alpha]+1)/2,\, m-([\alpha]+1)/2}(sv+sx)\,dx \\
&= \frac{s^{[\alpha]+1}}{\Gamma(1+[\alpha]-\alpha)} \int_x^\infty (u-x)^{[\alpha]-\alpha} \phi_{k+([\alpha]+1)/2,\, m-([\alpha]+1)/2}(su)\,du.
\end{aligned}
$$

But the right-hand side is an integral and so it is absolutely continuous as a function of x. Also when α is an integer $[\alpha]-\alpha = 0$ and hence

$$D^\alpha \phi_{k,m}(sx) = s^\alpha \phi_{k+\alpha/2,\, m-\alpha/2}(sx) = (-1)^\alpha (d/dx)^\alpha \phi_{k,m}(sx).$$

Lemma 3.3—The function

$$D^\alpha[(s/s_0)^{m-1/2} e^{-(s-s_0)x/2} W_{k,m}(sx)/W_{k,m}(s_0 x)]$$

is absolutely continuous provided $\alpha \geq 0$ and $m-(\alpha+1)/2 > 0$.

PROOF: By definition

$$
\begin{aligned}
&D^\alpha[(s/s_0)^{m-1/2} e^{-(s-s_0)x/2} W_{k,m}(sx)/W_{k,m}(s_0 x)] \\
&= \frac{(-1)^{[\alpha]+1}}{\Gamma(1+[\alpha]-\alpha)} (d/dx)^{[\alpha]} \lim_{w\to\infty} (d/dx) \int_x^w (u-x)^{[\alpha]-\alpha} (s/s_0)^{m-1/2} \\
&\qquad \times e^{-(s-s_0)u/2} \{W_{k,m}(su)/W_{k,m}(s_0 u)\}\,du.
\end{aligned}
$$

Now

$$(d/dx)\int_x^w (u-x)^{[\alpha]-\alpha}\, e^{-(s-s_0)u/2}\{W_{k,m}(su)/W_{k,m}(s_0u)\}\,du$$

$$= (d/dx)\Big\{[\{(u-x)^{1+[\alpha]-\alpha}/(1+[\alpha]-\alpha)\}\{e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}]_x^w$$

$$-\int_x^w \{(u-x)^{1+[\alpha]-\alpha}/(1+[\alpha]-\alpha)\}(d/du)\{e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}\,du\Big\}$$

$$= -(d/dx)\int_x^w \{(u-x)^{[\alpha]-\alpha+1}/([\alpha]-\alpha+1)\}(d/du)\{e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}\,du$$

$$+(d/dx)[\{(w-x)^{1+[\alpha]-\alpha}/(1+[\alpha]-\alpha\}\,e^{-(s-s_0)w/2}\,W_{k,m}(su)/W_{k,m}(s_0w)]$$

$$= \int_x^w (u-x)^{[\alpha]-\alpha}(d/du)\{e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}\,du$$

$$+(d/dx)[\{(w-x)^{1+[\alpha]-\alpha}/(1+[\alpha]-\alpha)\}\,e^{-(s-s_0)w/2}\,W_{k,m}(sw)/W_{k,m}(s_0w)]$$

$$= \int_x^w (u-x)^{[\alpha]-\alpha}(d/du)\{e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}\,du$$

$$-(w-x)^{[\alpha]-\alpha}\,e^{-(s-s_0)w/2}\,W_{k,m}(sw)/W_{k,m}(s_0w).$$

Therefore

$$(d/dx)^\alpha \lim_{w\to\infty}(d/dx)\int_x^w \{(u-x)^{[\alpha]-\alpha}\,e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}\,du$$

$$= (d/du)^\alpha \int_x^\infty (u-x)^{[\alpha]-\alpha}(d/du)\{e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}\,du$$

$$= (d/dx)^{[\alpha]}\int_0^\infty v^{[\alpha]-\alpha}(d/dv)\left\{e^{-(s-s_0)(v+x)/2}\frac{W_{k,m}(sv+sx)}{W_{k,m}(s_0v+s_0x)}\right\}dv$$

$$= \int_0^\infty v^{[\alpha]-\alpha}(d/dx)^{[\alpha]}\left\{e^{-(s-s_0)(v+x)/2}\frac{W_{k,m}(sv+sx)}{W_{k,m}(s_0v+s_0x)}\right\}dv$$

$$= \int_0^\infty v^{[\alpha]-\alpha}\{d/d(v+x)\}^{[\alpha]+1}\left\{e^{-(s-s_0)(v+x)/2}\frac{W_{k,m}(sv+sx)}{W_{k,m}(s_0v+s_0x)}\right\}dv$$

since $d(v+x)/dx = 1 = d(v+x)/dv$ and the function involved in differentiation is a function of $(v+x)$, this is

$$= \int_x^\infty (u-x)^{[\alpha]-\alpha}(d/du)^{[\alpha]+1}[e^{-(s-s_0)u/2}\{W_{k,m}(su)/W_{k,m}(s_0u)\}]\,du.$$

Hence

$$D^\alpha\{\phi_{k,m}(sx)/\phi_{k,m}(s_0x)\}$$

$$= D^\alpha\left\{(s/s_0)^{m-1/2}\,e^{-(s-s_0)x/2}\frac{W_{k,m}(sx)}{W_{k,m}(s_0x)}\right\}$$

$$= (s/s_0)^{m-1/2}\{(-1)^{[\alpha]+1}/\Gamma(1+[\alpha]-\alpha)\}$$

$$\times\int_x^\infty (u-x)^{[\alpha]-\alpha}(d/du)^{[\alpha]+1}\{e^{-(s-s_0)u/2}\,W_{k,m}(su)/W_{k,m}(s_0u)\}\,du.$$

But the right-hand side is an indefinite integral and hence absolutely continuous as a function of x. This establishes the lemma.

Lemma 3.4—For $t > 0$, $s = \sigma + i\tau$, $\sigma \geq 0$, k, m real

$$|\phi_{k,m}(st)| \leq (|s|/\sigma)^p |\phi_{k,m}(\sigma t)|,$$

where $p = k+m-1/2$ if $k+m-1/2 \geq 0$ and $p = 0$ if $k+m-1/2 \leq 0$.

PROOF: The Whittaker's function (Varma 1951) $W_{k,m}(z)$ is defined by the contour integral

$$(-1/2\pi i)\,\Gamma(k+(1/2)-m)\,e^{-z/2}\,z^k \int_{\infty}^{(0+)} (-t)^{-k+m-1/2}(1+t/z)^{k+m-1/2}\,e^{-t}\,dt \quad (3.2)$$

where $R(k-m-1/2) > 0$ and by the infinite integral

$$[e^{-z/2}\,z^k/\Gamma(-k+m+1/2)] \int_0^{\infty} t^{-k+m-1/2}(1+t/z)^{k+m-1/2}\,e^{-t}\,dt \quad .. \quad (3.3)$$

when $R(k-m-1/2) \leq 0$.

On transforming the contour integral (3.2) into the ordinary integral, (3.2) reduces to

$$[\{\sin(-k+m+1/2)\pi\}/\pi]\,\Gamma(k-m+1/2)\,e^{-z/2}\,z^k \int_0^{\infty} t^{m-k-1/2}(1+t/z)^{k+m-1/2}\,e^{-t}\,dt.$$

Hence

$$\phi_{k,m}(st) = [\{\sin(-k+m+1/2)\pi\}/\pi]\,\Gamma(k+m-1/2)\,e^{-st}(st)^{k+m-1/2} \times \int_0^{\infty} u^{m-k-1/2}(1+u/st)^{k+m-1/2}\,e^{-u}\,du$$

when $R(k-m-1/2) > 0$, and

$$= [1/\Gamma(m-k+1/2)]\,e^{-st}(st)^{k+m-1/2} \int_0^{\infty} u^{m-k-1/2}(1+u/st)^{k+m-1/2}\,e^{-u}\,du$$

when $R(k-m-1/2) \leq 0$.

It is clear from the above expression that $\phi_{k,m}(st)$ is always positive in the second case, while it has the same sign as $\sin(m-k+1/2)\pi$ in the first case and so the integrand involved in both these cases is positive when k and m are real.

We consider the two cases of this inequality separately.

Case (I): $k+m-1/2 \leq 0$.

$$\left| e^{-st}(st)^{k+m-1/2} \int_0^{\infty} u^{-k+m-1/2}(1+u/st)^{k+m-1/2}\,e^{-u}\,du \right|$$

$$= e^{-\sigma t} \left| \int_0^{\infty} u^{m-k-1/2}(u+st)^{k+m-1/2}\,e^{-u}\,du \right|$$

$$\leq e^{-\sigma t} \int_0^{\infty} u^{-k+m-1/2}\,|u+\sigma t|^{k+m-1/2}\,e^{-u}\,du.$$

The other factors being independent of s, we get

$$|\phi_{k,m}(st)| \leq |\phi_{k,m}(\sigma t)|. \quad .. \quad .. \quad .. \quad (3.4)$$

Case (II): $k+m-1/2 \geq 0$.

$$\left| e^{-st}(st)^{k+m-1/2} \int_0^\infty u^{m-k-1/2}(1+u/st)^{k+m-1/2}\, e^{-u}\, du \right|$$

$$\leq e^{-\sigma t}\,|st|^{k+m-1/2} \int_0^\infty u^{m-k-1/2}(1+u/\sigma t)^{k+m-1/2}\, e^{-u}\, du$$

$$= e^{-\sigma t}(|s|/\sigma)^{k+m-1/2}(\sigma t)^{k+m-1/2} \int_0^\infty u^{m-k-1/2}(1+u/\sigma t)^{k+m-1/2}\, e^{-u}\, du.$$

Therefore

$$|\phi_{k,m}(st)| \leq (|s|/\sigma)^{k+m-1/2}\,|\phi_{k,m}(\sigma t)|.$$

The two results (3.3) and (3.4) can finally be combined to give the following:

$$|\phi_{k,m}(st)| \leq (|s|/\sigma)^{p}\,|\phi_{k,m}(\sigma t)|,$$

where p is as defined in the statement of the lemma.

Lemma 3.5—If $\sigma \geq 0$, $A > 0$, $\alpha > 0$

$$\int_A^\infty x^\alpha\,|D^{\alpha+1}\phi_{k,m}(sx)|\,dx$$

is bounded provided $m-(\alpha+1)/2 > 0$.

PROOF—*Case (I)*: Let α be an integer, then

$$\int_A^\infty x^\alpha\,|D^{\alpha+1}\phi_{k,m}(sx)|\,dx = |s|^\alpha \int_A^\infty x^\alpha\,|\phi_{k+(\alpha+1)/2,\,m-(\alpha+1)/2}(sx)|\,dx$$

$$\leq (|s|/\sigma)^p\,|s|^\alpha \int_A^\infty x^\alpha\,|\phi_{k+(\alpha+1)/2,\,m-(\alpha+1)/2}(\sigma x)|\,dx$$

(By lemma 3.4, where p is as defined there)

$$= (|s|/\sigma)^{p+\alpha+1}\left| \int_{A\sigma}^\infty x^\alpha\,\phi_{k+(\alpha+1)/2,\,m-(\alpha+1)/2}(x)\,dx \right|$$

(The modulus sign has been taken out since the negative factor of the integrand, if any, can be taken out of the integral sign.)

$$\leq (|s|/\sigma)^{p+\alpha+1}\left| \int_0^\infty x^{m-1+\alpha/2}\, e^{-x/2}\, W_{k+(\alpha+1)/2,\,m-(\alpha+1)/2}(x)\,dx \right|$$

$$= (|s|/\sigma)^{p+\alpha+1}\,|\Gamma(\alpha+1)\Gamma(2m)/\Gamma(m-k-1/2)|$$

where the integral has been evaluated by applying Goldstein's integral (1932).

Case (II): Let α be not an integer, then on using lemma 3.2, we get

$$\int_A^\infty x^\alpha\,|D^{\alpha+1}\phi_{k,m}(sx)|\,dx$$

$$= \int_A^\infty x^\alpha\,|\{s^{[\alpha]+2}/\Gamma(1+[\alpha]-\alpha)\}\int_x^\infty (u-x)^{[\alpha]-\alpha}\, W_{k+1+[\alpha]/2,\,m-1-[\alpha]/2}(su)\,du\,dx$$

$$\leq \frac{|s|^{[\alpha]+2}}{\Gamma(1+[\alpha]-\alpha)}\,(|s|/\sigma)^p \int_A^\infty x^\alpha\,dx \int_x^\infty (u-x)^{[\alpha]-\alpha}\,|\phi_{k+1+[\alpha]/2,\,m-1-[\alpha]/2}(\sigma u)|\,du$$

$$= \frac{|s|^{[\alpha]+2}}{\Gamma(1+[\alpha]-\alpha)}\,(|s|/\sigma)^p \int_A^\infty |\phi_{k+1+[\alpha]/2,\,m-1-[\alpha]/2}(\sigma u)|\,du \int_A^u x^\alpha (u-x)^{[\alpha]-\alpha}\,dx$$

(The change in the order of integration can be justified by Fubini's theorem)

$$\leq \frac{|s|^{[\alpha]+2}}{\Gamma(1+[\alpha]-\alpha)}(|s|/\sigma)^p \int_A^\infty u^{[\alpha]+1}|\phi_{k+1-[\alpha]/2,\,m-1-[\alpha]/2}(\sigma u)\,du \int_0^1 v^\alpha(1-v)^{[\alpha]-\alpha}\,dv$$

$$= \{|s|^{[\alpha]+2}/\Gamma(1+[\alpha]-\alpha)\}(|s|/\sigma)^p B(\alpha+1,\,[\alpha]-\alpha+1)$$

$$\times \int_A^\infty u^{[\alpha]+1}|\phi_{k+1+[\alpha]/2,\,m-1-[\alpha]/2}(\sigma u)|\,du$$

($B(\alpha+1,\,[\alpha]-\alpha+1)$ is the Beta function)

$$= |s|^{[\alpha]+2}\{\Gamma(\alpha+1)/\Gamma([\alpha]+2)\}(|s|/\sigma)^p \left| \int_A^\infty u^{[\alpha]+1}\,\phi_{k+1+[\alpha]/2,\,m-1-[\alpha]/2}(\sigma u) \right| du$$

(The shifting of the modulus sign from inside the integral sign to outside can be explained as in case (I))

$$= \{\Gamma(\alpha+1)/\Gamma([\alpha]+2)\}(|s|/\sigma)^{p+[\alpha]+2} \left| \int_{A\sigma}^\infty v^{[\alpha]+1}\,\phi_{k+1+[\alpha]/2,\,m-1-[\alpha]/2}(v)\,dv \right|$$

$$\leq \{\Gamma(\alpha+1)/\Gamma([\alpha]+2)\}(|s|/\sigma)^{p+[\alpha]+2} \left| \int_0^\infty v^{[\alpha]+1}\,\phi_{k+1+[\alpha]/2,\,m-1-[\alpha]/2}(v)\,dv \right|$$

$$= \{\Gamma(\alpha+1)/\Gamma([\alpha]+2)\}(|s|/\sigma)^{p+[\alpha]+2}\,|\Gamma(2m)\Gamma([\alpha]+2)/\Gamma(m-k+1/2)|$$

$$= \Gamma(\alpha+1)(|s|/\sigma)^{p+[\alpha]+2}\,|\Gamma(2m)/(m-k+1/2)|$$

where the integral has been evaluated by Goldstein's formula.

The right-hand side being a constant depending on s and σ, the lemma is established.

Lemma 3.6—If $\sigma > \sigma_0 > 0$, $A > 0$, $\alpha > 0$,

$$\int_A^\infty x^\alpha\,|\,D^{\alpha+1}\{\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}\,|\,dx$$

is bounded provided $m-(\alpha+1)/2 > 0$.

PROOF:

$$\int_A^\infty x^\alpha\,|\,D^{\alpha+1}\{\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}\,|\,dx$$

$$= |\{(-1)^{[\alpha]+2}/\Gamma(1+[\alpha]-\alpha)\}(s/s_0)^{m-1/2}| \int_A^\infty x^\alpha\,dx$$

$$\times \left| \int_x^\infty (u-x)^{[\alpha]-\alpha}(d/du)^{[\alpha]+2}[e^{-(s-s_0)u/2}\,W_{k,\,m}(su)/W_{k,\,m}(s_0u)]\,du \right|$$

$$\leq \{1/|\Gamma(1+[\alpha]-\alpha)|\}\,|s/s_0|^{m-1/2} \int_A^\infty x^\alpha\,dx$$

$$\times \int_x^\infty (u-x)^{[\alpha]-\alpha}\,|\,(d/du)^{[\alpha]+2}[e^{-(s-s_0)u/2}\,W_{k,\,m}(su)/W_{k,\,m}(s_0u)]\,|\,du$$

$$= \frac{|s/s_0|^{m-1/2}}{|\Gamma(1+[\alpha]-\alpha)|} \int_A^\infty \left| (d/du)^{[\alpha]+2}\left[e^{-(s-s_0)u/2}\frac{W_{k,\,m}(su)}{W_{k,\,m}(s_0u)}\right] \right| du \int_A^u x^\alpha(u-x)^{[\alpha]-\alpha}dx$$

(The change in the order of integration is justifiable by Fubini's theorem)

$$< |s/s_0|^{m-1/2} |B(\alpha+1, [\alpha]-\alpha+1)/\Gamma(1+[\alpha]-\alpha)|$$

$$\times \int_A^\infty u^{[\alpha]+1} |(d/du)^{[\alpha]+2}[e^{-(s-s_0)u/2} W_{k,m}(su)/W_{k,m}(s_0u)]| du$$

$$= |s/s_0|^{m-1/2} |\Gamma(\alpha+1)/\Gamma([\alpha]+2)|$$

$$\times \int_A^\infty u^{[\alpha]+1} |(d/du)^{[\alpha]+2}[e^{-(s-s_0)u/2} W_{k,m}(su)/W_{k,m}(s_0u)]| du.$$

It is easy to see that the integral on the right-hand side exists and hence the given expression of the lemma is bounded by a constant which is a function of s, s_0, σ and σ_0.

PROOF OF THEOREM 1—*Case (I)*: Let $s_0 = 0$, so that we are required to prove that if $\int_0^\infty dA(u)$ is summable (C, α) [or $|C, \alpha|$], then the integral (1.1) is also summable (C, α) [or $|C, \alpha|$].

It is clear that $\phi_{k,m}(su)$ is $o(1)$ as $u \to \infty$. From lemma 3.2, $D^\alpha \phi_{k,m}(sx)$ is absolutely continuous and the remaining condition of lemma 3.1 is also seen to hold in lemma 3.5. Hence the theorem is established in this case.

Case (II): $s_0 \neq 0$. Since

$$C(u; s) = \int_0^u \phi_{k,m}(st)\, dA(t),$$

we have

$$\int_0^\infty \phi_{k,m}(su)\, dA(u) = \int_0^\infty \{\phi_{k,m}(su)/\phi_{k,m}(s_0u)\}\, dC(u; s_0).$$

Obviously $\phi_{k,m}(sx)/\phi_{k,m}(s_0x) = o(1)$ as $x \to \infty$. That $D^\alpha[\phi_{k,m}(sx)/\phi_{k,m}(s_0x)]$ is absolutely continuous as a function of x has been established in lemma 3.3. The remaining condition of lemma 3.1 has been established in lemma 3.6. Hence (1.1) is summable (C, α) [or $|C, \alpha|$] according as $C(u; s_0)$ as $u \to \infty$ or the integral (3.1) is summable (C, α) [or $|C, \alpha|$].

Thus the theorem is completely established.

Lemma 3.7—Let $\alpha > 0$ and $m-(h+1)/2 > 0$. Then

$$(d/dx)\left[x^{-\alpha} \int_0^x (x-t)^\alpha \phi_{k,m}(st)\, dA(t)\right]$$

$$= \Gamma(\alpha+1)\, x^{-\alpha-1}\, \phi_{k,m}(sx)\, \bar{A}_{\alpha+1}(x) + \{\Gamma(\alpha+1)/\Gamma(\alpha-h-1)\}\, x^{-\alpha-1}$$

$$\times \int_0^x \bar{A}_h(t)\{\phi_{k,m}(st) - \phi_{k,m}(sx)\}(x-t)^{\alpha-h-2}\, dt + x^{-\alpha-1} \sum_{r=1}^{h+1} s^r \binom{h+1}{r}$$

$$\times \{\Gamma(\alpha+1)/\Gamma(\alpha-h+r-1)\} \int_0^x \bar{A}_h(t)\, \phi_{k+r/2,\, m-r/2}(st)(x-1)^{\alpha-h+r-2}\, dt.$$

PROOF:

$$(d/dx)\left[x^{-\alpha}\int_0^x (x-t)^{\alpha}\,\phi_{k,\,m}(st)\,dA(t)\right]$$

$$= \alpha x^{-\alpha-1}\int_0^x (x-t)^{\alpha-1}\,\phi_{k,\,m}(st)\,t\,dA(t)$$

$$= \Gamma(\alpha+1)\,x^{-\alpha-1}\,\phi_{k,\,m}(sx)\,\bar{A}_{\alpha-1}(x)+(-1)^{h+1}\,\alpha x^{-\alpha-1}$$

$$\times\int_0^x \bar{A}_h(t)(d/dt)^{h+1}[\{\phi_{k,\,m}(st)-\phi_{k,\,m}(sx)\}(x-t)^{\alpha-1}]\,dt, \qquad (3.5)$$

by $(h+1)$ integration by parts.

Now

$$(d/dt)^{h+1}[\{\phi_{k,\,m}(st)-\phi_{k,\,m}(sx)\}(x-t)^{\alpha-1}]$$

$$= (-1)^{h+1}\left[\{\Gamma(\alpha)/\Gamma(\alpha-h-1)\}\{\phi_{k,\,m}(st)-\phi_{k,\,m}(sx)\}(x-t)^{\alpha-h-2}\right.$$

$$\left.+\sum_{r=1}^{h+1} s^r\binom{h+1}{r}\{\Gamma(\alpha)/\Gamma(\alpha-h+r-1)\}\,\phi_{k+r/2,\,m-r/2}(st)(x-t)^{\alpha-h+r-2}\right]. \qquad (3.6)$$

From (3.5) and (3.6), we get

$$(d/dx)\left[x^{-\alpha}\int_0^x (x-t)^{\alpha}\,\phi_{k,\,m}(st)\,dA(t)\right]$$

$$= \Gamma(\alpha+1)\,x^{-\alpha-1}\,\phi_{k,\,m}(sx)\,\bar{A}_{\alpha-1}(x)$$

$$+\{\Gamma(\alpha+1)/\Gamma(\alpha-h-1)\}\,x^{-\alpha-1}\int_0^x \bar{A}_h(t)\{\phi_{k,\,m}(st)-\phi_{k,\,m}(sx)\}(x-t)^{\alpha-h-2}\,dt$$

$$+x^{-\alpha-1}\sum_{r=1}^{h+1} s^r\binom{h+1}{r}\{\Gamma(\alpha+1)/\Gamma(\alpha-h+r-1)\}$$

$$\times\int_0^x (x-t)^{\alpha-h+r-2}\,\bar{A}_h(t)\,\phi_{k+r/2,\,m-r/2}(st)\,dt.$$

Lemma 3.8—If $\alpha > 0$, $\sigma > \sigma_0 > 0$, $m-(h+1)/2 > 0$, and if

$$B(t) = C(t;\,s_0) = \int_0^t \phi_{k,\,m}(s_0 u)\,dA(u),$$

then

$$(d/dx)\left[x^{-\alpha}\int_0^x (x-t)^{\alpha}\,\phi_{k,\,m}(st)\,dA(t)\right]$$

$$= \Gamma(\alpha+1)\,x^{-\alpha-1}\{\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}\,\bar{B}_{\alpha-1}(x)+\{\Gamma(\alpha+1)/\Gamma(\alpha-h-1)\}\,x^{-\alpha-1}$$

$$\times\int_0^x \bar{B}_h(t)\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)-\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}(x-t)^{\alpha-h-2}\,dt$$

$$+x^{-\alpha-1}\sum_{r=1}^{h+1}\binom{h+1}{r}\{\Gamma(\alpha+1)/\Gamma(\alpha-h+r-1)\}(-1)^r$$

$$\times\int_0^x \bar{B}_h(t)\,(d/dt)^r\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)\}(x-t)^{\alpha-h+r-2}\,dt.$$

PROOF:

$$(d/dx)\left[x^{-\alpha}\int_0^x (x-t)^{\alpha}\,\phi_{k,\,m}(st)\,dA(t)\right]$$

$$= (d/dx)\left[x^{-\alpha}\int_0^x (x-t)^{\alpha}\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)\}\,\phi_{k,\,m}(s_0t)\,dA(t)\right]$$

$$= (d/dx)\left[x^{-\alpha}\int_0^x (x-t)^{\alpha}\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)\}\,dB(t)\right]$$

$$= -\alpha x^{-\alpha-1}\int_0^x (x-t)^{\alpha}\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)\,dB(t)$$

$$+\alpha x^{-\alpha}\int_0^x (x-t)^{\alpha-1}\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)\,dB(t)$$

$$= \alpha x^{-\alpha-1}\int_0^x (x-t)^{\alpha-1}[\{\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}+\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)$$

$$-\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}]\,t\,dB(t)$$

$$= \Gamma(\alpha+1)\,x^{-\alpha-1}\{\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}\,\bar{B}_{\alpha-1}(x)$$

$$+\alpha x^{-\alpha-1}\int_0^x (x-t)^{\alpha-1}[\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)-\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)]\,t\,dB(t)$$

on performing $(h+1)$ integration by parts, we get this expression

$$= \Gamma(\alpha+1)\,x^{-\alpha-1}\{\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}\,\bar{B}_{\alpha-1}(x)$$

$$+(-1)^{h+1}\,\alpha x^{-\alpha-1}\int_0^x \bar{B}_h(t)(d/dt)^{h+1}[(x-1)^{\alpha-1}$$

$$\times\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)-\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)]\,dt. \qquad \ldots \quad (3.7)$$

On substituting the expression for

$$(d/dt)^{h+1}[\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)-\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}(x-t)^{\alpha-1}]$$

of the form of (3.6), we get the expression in (3.7)

$$= \Gamma(\alpha+1)\,x^{-\alpha-1}\{\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}\,\bar{B}_{\alpha-1}(x)$$

$$+\{\Gamma(\alpha+1)/\Gamma(\alpha-h-1)\}\,x^{-\alpha-1}\int_0^x \bar{B}_h(t)(x-t)^{\alpha-h-2}$$

$$\times\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)-\phi_{k,\,m}(sx)/\phi_{k,\,m}(s_0x)\}\,dt$$

$$+x^{-\alpha-1}\sum_{r=1}^{h+1}(-1)^r\{\Gamma(\alpha+1)/\Gamma(\alpha-h+r-1)\}$$

$$\times\int_0^x \bar{B}_h(t)(d/dt)^r\{\phi_{k,\,m}(st)/\phi_{k,\,m}(s_0t)\}(x-t)^{\alpha-h+r-2}\,dt.$$

This establishes the lemma.

The following lemma is immediate

Lemma 3.9—$1/\phi_{k,\,m}(z) = o(z^{-k-m+1/2}\,e^{z}), \quad (z\to\infty).$

Lemma 3.10 (Poussin (1923))—The nth derivative of $\{F(x)\}^m$ is the sum of a number of terms of the form

$$A\{F(x)\}^{m-r}\{F^1(x)\}^{\alpha_1}\{F^2(x)\}^{\alpha_2}\dots\{F^n(x)\}^{\alpha_n}$$

where A is a constant, $r \leq n$, and α's are positive integer or zeros such that

$$\sum_{v=1}^{n} a_v = r, \quad \sum_{v=1}^{n} v a_v = n.$$

Further, if m is a positive integer, then $0 < r \leq m$.

Lemma 3.11—$(d/dt)^n\{1/\phi_{k,m}(st)\} = o\{t^{-k-m+1/2}e^{st}\}, \quad (t \to \infty)$.

PROOF: From lemma 3.10, we get

$$\begin{aligned}(d/dt)^n\{1/\phi_{k,m}(st)\} &= \sum A\{\phi_{k,m}(st)\}^{-1-r}\{(-s)\,\phi_{k+1/2,m-1/2}(st)\}^{\alpha_1} \\ &\quad \times\{(-s)^2\phi_{k+1,m-1}(st)\}^{\alpha_2}\dots\{(-s)^n\phi_{k+n/2,m-n/2}(st)\}^{\alpha_n} \\ &= \sum A(-s)^{\sum_{v=1}^{n} v\alpha_v}\{\phi_{k,m}(st)\}^{-1-r} \\ &\quad \times\{\phi_{k+1/2,m-1/2}(st)\}^{\alpha_1}\dots\{\phi_{k+n/2,m-n/2}(st)\}^{\alpha_n}.\end{aligned}$$

Therefore

$$(d/dt)^n\{1/\phi_{k,m}(st)\} = (-s)^n \sum A \cdot o\left\{(st)^{\left(-1-r+\sum_{v=1}^{n}\alpha_v\right)(k+m-1/2)} \times e^{st\left(1+r-\sum_{v=1}^{n}\alpha_v\right)}\right\}, \quad (t \to \infty)$$

by use of lemmas 3.9 and 3.10

$$\begin{aligned}&= (-s)^n \sum A \cdot o\{(st)^{-k-m+1/2}\,e^{st}\}, \quad (t \to \infty) \\ &= o\{t^{-k-m+1/2}\,e^{st}\}, \quad (t \to \infty).\end{aligned}$$

This establishes the lemma.

Lemma 3.12—If $\sigma \geq 0$, $0 \leq t \leq u \leq x$, and $p = k+m-1/2$ if $k+m-1/2 \geq 0$ and $p = 0$ if $k+m-1/2 \leq 0$, then

$$\begin{aligned}|\phi_{k,m}(st)-\phi_{k,m}(sx)| &\leq (|s|/\sigma)^{p+1}\,|\phi_{k,m}(\sigma t)-\phi_{k,m}(\sigma x)| \\ &= (|s|/\sigma)^{p+1}\,\sigma(x-t)\,|\phi_{k+1/2,m-1/2}(\sigma u)|.\end{aligned}$$

PROOF:

$$\begin{aligned}|\phi_{k,m}(st)-\phi_{k,m}(sx)| &= \left|\int_t^x s\phi_{k+1/2,m-1/2}(su)\,du\right| \\ &\leq |s|\int_t^x |\phi_{k+1/2,m-1/2}(su)|\,du \\ &\leq (|s|/\sigma)^{p+1}\int_t^x |\phi_{k+1/2,m-1/2}(\sigma u)|\,d(\sigma u)\end{aligned}$$

(by lemma 3.4)

$$= (|s|/\sigma)^{p+1} \left| \int_t^x \phi_{k+1/2,\, m-1/2}(\sigma u)\, d(\sigma u) \right|$$

(The shifting of the modulus sign can be justified as in lemma 3.4)

$$= (|s|/\sigma)^{p+1} \left| [-\phi_{k,\, m}(\sigma u)]_t^x \right|$$

$$= (|s|/\sigma)^{p+1} |\phi_{k,\, m}(\sigma t) - \phi_{k,\, m}(\sigma x)|. \quad .. \quad .. \quad .. \quad (3.8)$$

By using mean-value theorem, the expression on the right of (3.8) is

$$= (|s|/\sigma)^p\, \sigma(x-t)\, |\phi_{k+1/2,\, m-1/2}(\sigma u)| \quad .. \quad .. \quad .. \quad (3.9)$$

where u is a suitable number satisfying the inequality $t \leq u \leq x$.
Thus the lemma is established in (3.8) and (3.9).

Lemma 3.13—If r is integral,

$$(d/dt)^r \{\phi_{k,\, m}(st)/\phi_{k,\, m}(s_0 t)\} = o\{(s/s_0)^{k+m-1/2} (s-s_0)^r\, e^{-(s-s_0)t}\}, \quad (t \to \infty).$$

A particular case of this result when $r = 1$ has been given by Singh (1956).

PROOF: From lemma 3.10, we get, when r is a positive integer,

$$(d/dt)^r [\phi_{k,\, m}(st)/\phi_{k,\, m}(s_0 t)]$$

$$= \sum_{u=1}^{r} \{r!/u!\,(r-u)!\} \phi_{k+(r-u)/2,\, m-(r-u)/2}(st)(-s)^{r-u} \Big[\sum A \{\phi_{k,\, m}(s_0 t)\}^{-1-v}$$

$$\times \{(-s_0)\, \phi_{k+1/2,\, m-1/2}(s_0 t)\}^{\alpha_1} \ldots \{(-s_0)^u\, \phi_{k+u/2,\, m-u/2}(s_0 t)\}^{\alpha_u} \Big]$$

$$\left(\text{where } \sum_{v=1}^{u} \alpha_v = v \text{ and } \sum_{v=1}^{u} v\alpha_v = u \right)$$

$$= \sum_{u=1}^{r} \binom{r}{u} \phi_{k+(r-u)/2,\, m-(r-u)/2}(st)(-s)^{r-u}(-s_0)^u \Big[\sum A \{\phi_{k,\, m}(s_0 t)\}^{-1-v}$$

$$\times \{\phi_{k+1/2,\, m-1/2}(s_0 t)\}^{\alpha_1} \ldots \{\phi_{k+u/2,\, m-u/2}(s_0 t)\}^{\alpha_u} \Big]$$

$$= \sum (-1)^r \binom{r}{u} s_0^u\, s^{r-u}\, o\{(st)^{k+m-1/2}\, e^{-st}\}\, o\Big[\sum o\{(s_0 t)^{-m-k+1/2}\, e^{s_0 t}\} \Big], \quad (t \to \infty)$$

$$= \sum (-1)^r \binom{r}{u} s_0^u\, s^{r-u} (s/s_0)^{k+m-1/2}\, o\{e^{-(s-s_0)t}\}, \quad (t \to \infty)$$

$$= o\{(s/s_0)^{k+m-1/2}\, e^{-(s-s_0)t}\}, \qquad (t \to \infty).$$

Hence there exists a constant N, depending on s and s_0, such that

$$|(d/dt)^r \{\phi_{k,\, m}(st)/\phi_{k,\, m}(s_0 t)\}| < N\, |s/s_0|^{k+m-1/2}\, e^{-(\sigma-\sigma_0)t} \quad .. \quad (3.10)$$

for large values of t.

By r-integrations of the order of

$$(d/dt)^r [\phi_{k,\, m}(st)/\phi_{k,\, m}(s_0 t)],$$

we get

$$o[\phi_{k,\, m}(st)/\phi_{k,\, m}(s_0 t)] = o[(s-s_0)^{-r} (s/s_0)^{k+m-1/2}\, e^{-(s-s_0)t}], \quad (t \to \infty). \quad (3.11)$$

By ordinary calculation, we get

$$o[\phi_{k,m}(st)/\phi_{k,m}(s_0t)] = o(s/s_0)^{k+m-1/2}\, e^{-(\sigma-\sigma_0)t} \quad .. \quad .. \quad (3.12)$$

The results (3.11) and (3.12) suggest that the constant N of (3.10) may be replaced by $(s-s_0)^r M$, where M is another constant which is not divisible by $(s-s_0)$.

Hence

$$|(d/dt)^r\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}| < M\,|s/s_0|^{m+k-1/2}\,|s-s_0|^r\, e^{-(\sigma-\sigma_0)t}$$

for large values of t, i.e.

$$(d/dt)^r\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\} = o[(s/s_0)^{k+m-1/2}(s-s_0)^r\, e^{-(s-s_0)t}], \quad (t\to\infty)$$

when r is a negative integer, the result can be arrived at by integration of (3.12).

Lemma 3.14—If $\sigma > \sigma_0$, $x > t \geq 0$, then there exists a constant N such that

$$|\phi_{k,m}(st)/\phi_{k,m}(s_0t) - \phi_{k,m}(sx)/\phi_{k,m}(s_0x)| < N(x-t)\,|(d/dt)\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}|.$$

PROOF: Since $\phi_{k,m}(su)/\phi_{k,m}(s_0u)$ is continuous and differentiable in $0 \leq u < U$, for any finite U, we have by mean-value theorem

$$|\phi_{k,m}(st)/\phi_{k,m}(s_0t) - \phi_{k,m}(sx)/\phi_{k,m}(s_0x)|$$
$$= (x-t)\,|(d/du)\{\phi_{k,m}(su)/\phi_{k,m}(s_0u)\}|, \quad (t \leq u \leq x). \quad .. \quad (3.13)$$

Also since

$$|(d/du)\,\phi_{k,m}(su)/\phi_{k,m}(s_0u)|\,/\,|(d/dt)\phi_{k,m}(st)/\phi_{k,m}(s_0t)| = o(1), \quad (t \leq u,\ t\to\infty),$$

there exists a constant M such that

$$|(d/du)\,\phi_{k,m}(su)/\phi_{k,m}(s_0u)| \leq M\,|(d/dt)\,\phi_{k,m}(st)/\phi_{k,m}(s_0t)|, \quad (0 \leq t \leq u). \quad .. \quad (3.14)$$

From (3.13) and (3.14) the lemma follows.

PROOF OF THEOREM 2—*Case* (*i*): $s_0 = 0$, i.e. we assume the summability (C, α) of

$$\int_0^\infty dA(t) \quad .. \quad .. \quad .. \quad .. \quad (3.15)$$

and prove that the integral (1.1) is summable $|C, \alpha+1|$ for $\sigma > 0$. Replacing α by $\alpha-1$, $\alpha \geq 1$, the summability $(C, \alpha-1)$ of (3.15) implies, on account of (2.1),

$$\bar{A}_{\alpha-1}(x) = o(x^\alpha), \quad (x\to\infty). \quad .. \quad .. \quad .. \quad (3.16)$$

To prove the theorem, let us write the expression on the right of lemma 3.7 as $\sum_{r=1}^{h+3} I_r$ and then it is sufficient to show that for each $r = 1, 2, 3, \ldots, h+3$; and $\sigma > 0$

$$\int^\infty |I_r|\,dx < \infty.$$

Firstly,

$$\int^{\infty} |I_1|\,dx \leq \Gamma(\alpha+1)\int^{\infty} |\bar{A}_{\alpha-1}(x)|\,x^{-\alpha-1}\,|\phi_{k,m}(sx)|\,dx$$

$$\leq \Gamma(\alpha+1)(|s|/\sigma)^p \int^{\infty} |\bar{A}_{\alpha-1}(x)|\,x^{-\alpha-1}\,|\phi_{k,m}(\sigma x)|\,dx$$

where p is as defined in lemma 3.12.
By use of (2.1) and order property of $\phi_{k,m}(\sigma t)$ it is obvious that the integral on the right exists for any positive lower limit.
Secondly,

$$\int^{\infty} |I_2|\,dx = c\int^{\infty} \left| x^{-\alpha-1}\int_0^x \bar{A}_h(t)[\phi_{k,m}(st)-\phi_{k,m}(sx)](x-t)^{\alpha-h-2}\,dt \right| dx$$

$$\leq c_1\int^{\infty} x^{-\alpha-1}\,dx\int_0^x |\bar{A}_h(t)|\,.\,|\phi_{k,m}(st)-\phi_{k,m}(sx)|\,(x-t)^{\alpha-h-2}\,dt$$

$$\leq c_2\int^{\infty} x^{-\alpha-1}\int_0^x |\bar{A}_h(t)|\,.\,|\phi_{k+1/2,\,m-1/2}(\sigma t)|\,(x-t)^{\alpha-h-1}\,dt$$

(by lemma 3.12)

$$\leq c_3\int^{\infty} |\bar{A}_h(t)|\,.\,|\phi_{k+1/2,\,m-1/2}(\sigma t)|\,dt\int_t^{\infty} x^{-\alpha-1}(x-t)^{\alpha-h-1}\,dx$$

$$= c_4\int^{\infty} |\bar{A}_h(t)|\,.\,|\phi_{k+1/2,\,m-1/2}(\sigma t)|\,t^{-h-1}\,dt < \infty$$

since $\bar{A}_h(t) = o(t^{h+1})$, $t \to \infty$. The c's involved are as defined in section 2 and change in the order of integration is justifiable by Fubini's theorem.
Finally,

$$\int^{\infty} |I_{r+2}|\,dx \leq c\int^{\infty} x^{-\alpha-1}\,dx\int_0^x |\bar{A}_h(t)|\,.\,|\phi_{k+r/2,\,m-r/2}(st)|\,(x-t)^{\alpha-h+r-2}\,dt$$

$$\leq c_1\int^{\infty} x^{-\alpha-1}\,dx\int_0^x |\bar{A}_h(t)|\,.\,|\phi_{k+r/2,\,m-r/2}(\sigma t)|\,(x-t)^{\alpha-h+r-2}\,dt$$

$$\leq c_2\int^{\infty} |\bar{A}_h(t)|\,.\,|\phi_{k+r/2,\,m-r/2}(\sigma t)|\,dt\int_t^{\infty} (x-t)^{\alpha-h+r-2}\,x^{-\alpha-1}\,dx$$

$$\leq c_3\int^{\infty} |\bar{A}_h(t)|\,.\,|\phi_{k+r/2,\,m-r/2}(\sigma t)|\,.\,t^{-h+r-2}\,dt < \infty$$

since $\bar{A}_h(t) = o(t^{h+1})$. The change in the order of integration is justifiable by Fubini's theorem.

Case (ii): $s_0 \neq 0$. In this case also we make the same assumption as in case (i), but use the expressions from lemma 3.8 instead of lemma 3.7 and $\bar{B}_{\alpha-1}(t) = o(x^{\alpha})$ as $x \to \infty$ instead of (3.16).
Now firstly,

$$\int^{\infty} |I_1|\,dx \leq \Gamma(\alpha+1)\int^{\infty} |\bar{B}_{\alpha-1}(x)|\,.\,x^{-\alpha-1}\,.\,|\phi_{k,m}(sx)/\phi_{k,m}(s_0x)|\,dx < \infty$$

since the integrand is $o\{x^{-1}\,e^{-(\sigma-\sigma_0)x}\}$ as $x \to \infty$.

Secondly,

$$\int^{\infty} |I_2|\,dx = c\int^{\infty} \Big| x^{-\alpha-1}\int_0^x \bar{B}_h(t)[\phi_{k,m}(st)/\phi_{k,m}(s_0t) - \phi_{k,m}(sx)/\phi_{k,m}(s_0x)] \times (x-t)^{\alpha-h-2}\,dt \Big|\,dx$$

$$\leq c_1\int^{\infty} x^{-\alpha-1}\,dx \int_0^x |\bar{B}_h(t)|\,.\,|\phi_{k,m}(st)/\phi_{k,m}(s_0t) - \phi_{k,m}(sx)/\phi_{k,m}(s_0x)| \times (x-t)^{\alpha-h-2}\,dt$$

$$\leq c_2\int^{\infty} x^{-\alpha-1}\,dx \int_0^x |\bar{B}_h(t)|\,.\,|(d/dt)\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}|\,(x-t)^{\alpha-h-1}\,dt$$

$$\leq c_3\int^{\infty} |\bar{B}_h(t)|\,.\,|(d/dt)\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}|\,dt \int_t^{\infty} x^{-\alpha-1}(x-t)^{\alpha-h-1}\,dx$$

(The change in the order of integration is justifiable by Fubini's theorem.)

$$\leq c_4\int^{\infty} |\bar{B}_h(t)|\,.\,t^{-h-1}\,.\,|(d/dt)\,\phi_{k,m}(st)/\phi_{k,m}(s_0t)|\,dt < \infty$$

since $\bar{B}_h(t) = o(t^{h+1})$ as $t \to \infty$. Finally, for $r > 0$,

$$\int^{\infty} |I_{r+2}|\,dx \leq c\int^{\infty} x^{-\alpha-1}\,dx \Big| \int_0^x \bar{B}_h(t)(d/dt)^r\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}(x-t)^{\alpha-h+r-2}\,dt \Big|$$

$$\leq c_1\int^{\infty} x^{-\alpha-1}\,dx \int_0^x |\bar{B}_h(t)|\,.\,|(d/dt)^r\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}|\,(x-t)^{\alpha-h+r-2}\,dt$$

$$\leq c_2\int^{\infty} |\bar{B}_h(t)|\,.\,|(d/dt)^r\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}|\,dt \int_t^{\infty} x^{-\alpha-1}(x-t)^{\alpha-h+r-2}\,dx$$

$$\leq c_3\int^{\infty} t^{-h+r-2}|\bar{B}_h(t)|\,.\,|(d/dt)^r\{\phi_{k,m}(st)/\phi_{k,m}(s_0t)\}|\,dt < \infty$$

by virtue of lemma 3.13 and the fact that $|\bar{B}_h(t)| = o(t^{h+1})$ as $t \to \infty$. Hence the theorem is established.

Corollary—If σ_α and $\bar{\sigma}_\alpha$ are the abscissa of summability (C, α) and $|C, \alpha|$ respectively, then

$$\sigma_\alpha \geq \bar{\sigma}_{\alpha+1} \geq \sigma_{\alpha+1}.$$

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