TRANSFORMATION FORMULAE FOR KAMPÉ DE FÉRIET HYPERGEOMETRIC FUNCTION

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In this paper, starting from Macrobert's result we have obtained two transformations for Kampé de Fériet double hypergeometric series and Saalschützian, Dixon and Watson types sums of double hypergeometric series.

§ 1. Carlitz (1967), Pandey and Saran (1963) and Singal (*in press a*) gave a number of transformation formulae of Kampé de Fériet hypergeometric function of order two. The object of this paper is to study the transformation of Kampé de Fériet function of higher order.

Let Kampé de Fériet function (Appel and Kampé de Fériet 1926) of order $r+s$ in a modified form be defined as

$$F^{p,\,q}_{r,\,s}\left[\begin{matrix}a_p : b_q ;\ b'_q ; \\ c_r\ : d_s ;\ d'_s ;\end{matrix} X,\ Y\right] = \sum_{m,\,n=0}^{\infty} \frac{(a_p)_{m+n}(b_q)_m(b'_q)_n X^m Y^n}{(c_r)_{m+n}(d_s)_m(d'_s)_n m!\ n!}$$

which is absolutely convergent if
(i) $p+q < r+s+1$, for all X and Y;
(ii) $p+q = r+s+1$, $p \neq r$, for all values of X and Y in the region common to $|X|^{\frac{1}{p-r}}+|Y|^{\frac{1}{p-r}} = 1$, and $|X| < 1$, $|Y| < 1$, containing origin; and
(iii) $p+q = r+s+1$, $p = r$, for $|X| < 1$, $|Y| < 1$;
where a_p stands for the sequence $a_1, a_2, \ldots, a_p$ and $(a)_m = a(a+1)(a+2)\ldots(a+m-1)$, $(a)_0 = 1$.

The main results to be proved are

$$F^{1,\,3}_{1,\,2}\left[\begin{matrix}d: & a,\ a';\ b,\ b' & ; & c,\ c' & ; \\ e: & 1+a-b,\ 1+a'-b'; & & 1+a-c,\ 1+a'-c'; & \end{matrix}\right] = \frac{\Gamma(e)\,\Gamma(e-a-a'-d)}{\Gamma(e-a-a')\,\Gamma(e-d)}$$

$$\times F^{2,\,3}_{2,\,2}\left[\begin{matrix}d,\ 1+a+a'-e\ :\ a/2,\ a'/2;\ (a+1)/2, & (a'+1)/2 ; & 1+a-b-c,\ 1+a'-b'-c'; \\ (1+a+a'-e+d)/2,\ (2+a+a'+d-e)/2 : 1+a-b, & 1+a'-b'; & 1+a-c\ ,\ \ 1+a'-c'\ ;\end{matrix}\right] \quad \ldots \quad (1.1)$$

provided d is a negative integer or a, a', are negative integers and

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$$F^{4,3}_{3,2}\left[\begin{matrix}-m\ , & d,\ d'\ , & 1+(a+a')/2 & : & a,\ a';\ b,\ b' & ; & c,\ c' & ; \\ 1+a+a'+m, & 1+a+a'-d, & 1+a+a'-d', & (a+a')/2: & 1+a-b,\ 1+a'-b'; & & 1+a-c,\ 1+a'-c'; & \end{matrix}\right]$$

$$=\frac{(1+a+a')_m(1+a+a'-d-d')_m}{(1+a+a'-d)_m(1+a+a'-d')_m}$$

$$\times F^{3,3}_{3,2}\left[\begin{matrix}-m & , & d,\ d' & : & a/2,\ a'/2 & ; & (a+1)/2,\ (a'+1)2 & ; & 1+a-b-c,\ 1+a'-b'-c'; \\ d+d'-a-a'-m, & & (a+a')/2,\ (a+a'+1)/2: & & 1+a-b,\ 1+a'-b'; & & 1+a-c,\ 1+a'-c' & ; & \end{matrix}\right]. \quad \ldots (1.2)$$

PROOF: Starting from Macrobert's result (1939)

$${}_4F_3\left[\begin{matrix}a,\ b,\ c,\ d & ; \\ 1+a-b,\ 1+a-c,\ e; & \end{matrix} X\right]=\sum_{n=0}^{\infty}\frac{(a)_{2n}(1+a-b-c)_n(d)_n(-X)^n}{n!\ (1+a-b)_n(1+a-c)_n(e)_n}\,{}_2F_1\left[\begin{matrix}a+2n,\ d+n\ ; \\ e+n\ ; \end{matrix} X\right]$$

it is easy to prove

$$F^{1,3}_{1,2}\left[\begin{matrix}d: & a,\ a';\ b,\ b' & ; & c,\ c' & ; \\ e: & 1+a-b,\ 1+a'-b'; & & 1+a-c,\ 1+a'-c'; & \end{matrix} X,\ Y\right]$$

$$=\sum_{n,N=0}^{\infty}\frac{(d)_{n+N}(a)_{2N}(a')_{2n}(1+a-b-c)_N(1+a'-b'-c')_n(-X)^N(-Y)^n}{(e)_{n+N}(1+a-b)_N(1+a'-b')_n(1+a-c)_N(1+a'-c')_nN!\ n!}$$

$$\times F_1(d+n+N:a+2N,\ a'+2n;\ e+n+N;\ X,\ Y). \quad \ldots \quad \ldots (1.3)$$

Right-hand side can be summed at $X = Y = 1$ if either d is a negative integer or a, a' are negative integers and hence we get (1.1).
Consider

$$\int_0^1 t^{a_p-1}(1-t)^{c_r-a_p-1}F^{p-1,q}_{r-1,s}\left[\begin{matrix}a_{p-1}:b_q,\ b'_q; \\ c_{r-1}\ :d_s,\ d'_s;\end{matrix} Xt,\ Yt\right]dt$$

$$=\frac{\Gamma(a_p)\,\Gamma(c_r-a_p)}{\Gamma(c_r)}F^{p,q}_{r,s}\left[\begin{matrix}a_p:b_q,\ b'_q; \\ c_r\ :d_s,\ d'_s;\end{matrix} X,\ Y\right] \quad \ldots \quad \ldots (1.4)$$

provided $c_r > a_p$, which can be proved by expansion and term by term integration.
Using (1.3) in (1.4) and after a little simplification we get

$$F^{p,3}_{p,2}\left[\begin{matrix}d_p: & a,\ a';\ b,\ b' & ; & c,\ c' & ; \\ e_p: & 1+a-b,\ 1+a'-b'; & & 1+a-c,\ 1+a'-c'; & \end{matrix} X,\ X\right]$$

$$=\sum_{n,N=0}^{\infty}\frac{(d_p)_{n+N}(a)_{2N}(a')_{2n}(1+a-b-c)_N(1+a'-b'-c')_n(-X)^{n+N}}{(e_p)_{n+N}(1+a-b)_N(1+a'-b')_n(1+a-c)_N(1+a'-c')_nN!\ n!}$$

$$\times {}_{p+1}F_p(d_p+n+N,\ a+a'+2n+2N;\ e_p+n+N;\ X). \quad \ldots \quad \ldots (1.5)$$

The series on the right can be summed at $X = 1$ if one of the d's is a negative integer and $p = 4$.

Taking $p = 4$, specializing the parameters and using the result (Bailey 1935, 4.3 (3))

$${}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, c, & -m; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a+m; \end{matrix}\right] = \frac{(1+a)_m(1+a-b-c)_m}{(1+a-c)_m(1+a-b)_m}$$

we get (1.2).

§ 2. Particular cases—Letting $m \to \infty$, (1.2) reduces to

$$F^{3,3}_{3,2}\left[\begin{matrix} d, d', & 1+(a+a')/2 & : a, a'; b, b' & ; c, c' & ; \\ 1+a+a'-d, 1+a+a'-d', & (a+a')/2 & : 1+a-b, 1+a'-b'; & 1+a-c, 1+a'-c'; \end{matrix} -1, -1\right]$$

$$= \Gamma(1+a+a'-d)\,\Gamma(1+a+a'-d')/\Gamma(1+a+a')\,\Gamma(1+a+a'-d-d')$$

$$\times F^{2,3}_{2,2}\left[\begin{matrix} d, d' & : a/2, a'/2 & ; (a+1)/2, (a'+1)/2; & 1+a-b-c, 1+a'-b'-c'; \\ (a+a')/2, (a+a'+1)/2 & : 1+a-b, 1+a'-b'; & & 1+a-c, 1+a'-c' ; \end{matrix} 1, 1\right]. \quad \text{.. (2.1)}$$

By proper choice of parameters, a number of transformations can be obtained from (1.1), (1.2) and (2.1).

Taking $a' = 0$, (1.1), (1.2) and (2.1) reduce to results (2; 4.5 (1) and 4.7 (1)), (2; 4.3 (4)) and (2; 4.4 (2)) respectively and other particular cases of (1.2) reduce to corresponding cases of Bailey (1935, 4.3 (4)).

§ 3. In this section we give some results analogous to Bailey's results (1935, p. 87).

If

$$F_3(b, b'; c, c'; e; X, Y)F_3(b, b'; c, c'; d; X, Y) = \sum_{m,n=0}^{\infty} A_{m,n}X^mY^n$$

then

$$F^{2,2}_{2,1}\left[\begin{matrix} (d+e)/2, (d+e-1)/2 : & b, b'; c, c'; \\ d, e & : b+c, b'+c'; \end{matrix} 4X(1-X), 4Y(1-Y)\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(d+e-1)_{m+n}}{(b+c)_m(b'+c')_n} A_{m,n}X^mY^n \quad \text{.. .. .. .. (3.1)}$$

and if

$$F_2(d; b, b'; c, c'; X, Y)F_2(e; b, b'; c, c'; X, Y) = \sum_{m,n=0}^{\infty} A_{m,n}X^mY^n$$

then

$$(1-X)^{-b}(1-Y)^{-b'}F^{2,2}_{2,1}\left[\begin{matrix} d, e & : c-b, c'-b'; b, b'; \\ (d+e)/2, (d+e-1)/2 : & c, c' & ; \end{matrix} \frac{-X^2}{4(1-X)}, \frac{-Y^2}{4(1-Y)}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(c)_m(c')_n}{(d+e)_{m+n}} A_{m,n}X^mY^n. \quad \text{.. .. .. (3.2)}$$

These results are actually equivalent to identity (1.1) when a, a' are negative integers.

§ 4. In the end we obtain certain sums which can be called Saalschützian, Dixon and Watson sums for double hypergeometric series.

Putting $c = \frac{1}{2}(1+a)$, $c' = \frac{1}{2}(1+a')$ and $e = 1+a+a'-d$ in (1.1) we get

$$F^{1,2}_{1,1}\left[\begin{matrix} d & : & a, a'; b, b' & ; \\ 1+a+a'-d & : & 1+a-b,\ 1+a'-b'; & \end{matrix}\right] = \frac{\Gamma(1+a+a'-d)\,\Gamma(1-2d)}{\Gamma(1-d)\,\Gamma(1+a+a'-2d)}$$

$$\times F^{1,2}_{1,1}\left[\begin{matrix} d : & a/2, a'/2 & ; (1+a-2b)/2, (1+a'-2b')/2 ; \\ \frac{1}{2}+d : & 1+a-b,\ 1+a'-b'; & \end{matrix}\right] \quad .. \quad (4.1)$$

provided either d or $a/2$, $a'/2$ are negative integers.

Now if $2d = 1+a+a'-2b-2b'$, the right side of (4.1) can be summed by the formula

$$F^{1,2}_{1,1}\left[\begin{matrix} a ; & b, b'; & c, c'; \\ d : & e, e'; & \end{matrix}\right] = \frac{\Gamma(b-a)\,\Gamma(b'-a)\,\Gamma(b-c')\,\Gamma(b'-c)\,\Gamma(d)\,\Gamma(d-a-c-c')}{\Gamma(b)\,\Gamma(b')\,\Gamma(b-a-c')\,\Gamma(b'-a-c)\,\Gamma(d-a)\,\Gamma(d-c-c')} \quad .. \quad (4.2)$$

provided $d = b+b'$, $d+e = 1+a+b+c$, $d+e' = 1+a+b'+c'$ and either a or c, c' are negative integers.

(4.2) can be easily proved by using the technique of Carlitz (1965) and can be called Saalschützian sum for the double series. In fact (4.2) is (7) of Carlitz (1965) or (9) of Carlitz (1963) according as a or c, c' are negative integers.

Hence (4.1) gives

$$F^{1,2}_{1,1}\left[\begin{matrix} d & : & a, a'; b, b' & ; \\ 1+a+a'-d & : & 1+a-b,\ 1+a'-b'; & \end{matrix}\right] = \frac{\Gamma(1+a+a'-d)\,\Gamma(1-2d)\,\Gamma(\frac{1}{2}+d)\,\Gamma((1-a-a')/2)}{\Gamma(1-d)\,\Gamma(1+a+a'-2d)\,\Gamma(\frac{1}{2})\,\Gamma((1+2d-a-a')/2)}$$

$$\times \frac{\Gamma((1+a-2b-2d)/2)\,\Gamma((1+a'-2b'-2d)/2)\,\Gamma((1+a-a'-2b)/2)\,\Gamma((1+a'-a-2b')/2)}{\Gamma((1+a-2b)/2)\,\Gamma((1+a'-2b')/2)\,\Gamma((1+a-a'-2b-2d)/2)\,\Gamma((1+a'-a-2b'-2d)/2)} \quad .. \quad (4.3)$$

provided $2d = 1+a+a'-2b-2b'$ and either d or $a/2$, $a'/2$ are negative integers. It may be called Dixon sum of double series.

To obtain Watson type sum, consider (2.1) of Singal (*in press b*)

$$F^{1,2}_{1,1}\left[\begin{matrix} a : & 2b, 2b' ; 2c, 2c' & ; \\ d : & b+c+\frac{1}{2},\ b'+c'+\frac{1}{2}; & \end{matrix}\right] = F^{2,2}_{2,1}\left[\begin{matrix} a, d-a & : & b, b' ; c, c' & ; \\ d/2, (d+1)/2 & : & b+c+\frac{1}{2},\ b'+c'+\frac{1}{2}; & \end{matrix}\right].$$

Taking $d = 2a$ and applying (4.2) on the right-hand side we get

$$F^{1,2}_{1,1}\left[\begin{matrix} a : & 2b, 2b' ; 2c, 2c' & ; \\ 2a : & b+c+\frac{1}{2},\ b'+c'+\frac{1}{2} ; & \end{matrix}\right] = \frac{\Gamma(b-a)\,\Gamma(b'-a)\,\Gamma(b-c')\,\Gamma(b'-c)\,\Gamma(a+\frac{1}{2})\,\Gamma(\frac{1}{2}-c-c')}{\Gamma(b)\,\Gamma(b')\,\Gamma(b-a-c')\,\Gamma(b'-a-c)\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+a-c-c')} \quad .. \quad (4.4)$$

provided $a+\frac{1}{2} = b+b'$ and either a or c, c' are negative integers.

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