ON THE LIMITING OSCILLATIONS OF A SOUNDING ROCKET EXITING THE ATMOSPHERE

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The paper studies the limiting motion of a coasting rocket vehicle. For the purpose certain properties of the solutions of a second order linear differential equation provide the basis of the study. Certain stability criteria relating to oscillatory motion of a sounding rocket have been generalized, and a new solution to Murphy's equation has been obtained.

1. INTRODUCTION

The stability of a coasting rocket ascending rapidly through the earth's atmosphere is a well-investigated problem in astronautics. For definiteness, references may be made to some of the recent works of Murphy (1963, 1965), Nicolaides (1959), Stone (1966) and others*. In the present investigation we avoid questions on roll resonance and consider certain points relating to the effects of a gradient in dynamic pressure on the oscillatory motion of a coasting rocket vehicle. For the purpose we have considered the well-known dynamical equations of Murphy (1963) characterizing linear oscillations of a missile in pitch and yaw. From known results in the theory of linear differential equations it may be proved that Murphy's equations will always lead to an oscillatory motion of the missile in the upper regions of the earth's atmosphere under certain well-defined conditions.

2. MURPHY'S EQUATIONS

Consider the dynamical equation due to Murphy (1963) of a coasting rocket referred to a non-rolling (aeroballistics) axis system in the form:

$$\xi'' + (\rho D - iG)\,\xi' - \rho(M + iGT)\,\xi = 0, \quad \left({}' = \frac{d}{ds}\right) \qquad (2.1)$$

where†

$$D = (A/2m)\{C_{N\alpha} - 2C_D' - k_t^{-2}(C_{Mq} + C_{M\dot{\alpha}})\} \qquad (2.2)$$

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* Friedrich and Dore (1955), Leon (1958), Garber (1959), Platus (1967), Vaughn (1968), Pettus (1966).

† For detailed explanation of the symbols used, *see* Stone (1966).

is known as the damping parameter,

$$G = (p/v)(I_x/I_y) \quad .. \quad .. \quad .. \quad .. \quad (2.3)$$

is the gyroscopic parameter,

$$M = Al\, C_{M\alpha}/2m \quad .. \quad .. \quad .. \quad .. \quad (2.4)$$

is the static moment parameter, and

$$T = (A/2m)(C_{N\alpha} - C_D + k_\alpha^{-2} C_{M_{p\alpha}}) \quad .. \quad .. \quad (2.5)$$

is the Magnus moment parameter.

The various aerodynamic parameters characterized by D, M and T are known to be very slowly varying functions of the path length s of the rocket, but in the present analysis they are assumed to be constants. If, however, we assume that the roll rate p is proportional to the velocity v of the missile, this will make the gyroscopic parameter G constant during the motion.

We further have

$$\rho = \rho_0 e^{-\beta h} \quad .. \quad .. \quad .. \quad .. \quad (2.6)$$

as the law of decay of atmospheric density as a function of the altitude h above some reference point and ρ_0 and β are constants.

If we assume that the average flight path can be approximated by straight line segments such that

$$h = s \sin \psi \quad .. \quad .. \quad .. \quad .. \quad (2.7)$$

where $\psi =$ constant is the average flight path angle with respect to a local horizontal, from (2.6) and (2.7) we have

$$\rho = \rho_0 e^{-ks} \quad .. \quad .. \quad .. \quad .. \quad (2.8)$$

where

$$\beta \sin \psi = k \text{ (constant)}.$$

It may be noted that ξ is a complex function of the real variable s and characterizes the vector yaw of the rocket, the aerodynamic derivatives C's occurring in (2.2), (2.4) and (2.5) have their usual meanings. I_x and I_y occurring in (2.3) are the axial and cross-axial moments of inertia of the rocket, and k_α and k_t, the respective radii of gyration (suitably non-dimensionalized) and are given by

$$\left.\begin{aligned} k_\alpha &= (I_x/ml^2)^{1/2} \\ k_t &= (I_y/ml^2)^{1/2} \end{aligned}\right\}. \quad .. \quad .. \quad .. \quad .. \quad (2.9)$$

3. A Perturbation Solution of Murphy's Equation

Closed solution of (2.1) in terms of confluent hypergeometric functions and asymptotic solutions by W.B.K.J. approximation have been considered by Stone (1966). The perturbation solution we intend to discuss here will be found useful to make quantitative estimates of the limiting oscillations of the missile. If we put

$$\xi = u \exp\{1/2(\rho D/k)+iGs/2\} \quad \ldots \quad (3.1)$$

(2.1) may be transformed to

$$u''+I(s, \rho)u = 0, \; ('=d/ds) \quad \ldots \quad (3.2a)$$

where

$$I(s, \rho) = (G^2/4)+\rho\{(kD-M)-iGT+(iGD/2)\}-\rho^2D^2/4. \quad \ldots \quad (3.2b)$$

Now we seek a solution of (3.2) in the form

$$u(s) = w(s) \exp\{\phi(s, \rho)\}. \quad \ldots \quad (3.3)$$

Substituting (3.3) in (3.2) and using (2.8), we readily obtain the recursive equations in the form

$$w''+(G^2/4)w = 0 \quad \ldots \quad (3.4)$$

and

$$\phi''+\left(\frac{2w'}{w}\right)\phi'+\phi'^2+\rho\{(kD-M)-iGT+(iGD/2)\}-\rho^2D^2/4 = 0. \quad \ldots \quad (3.5)$$

Since G is real, taking the solution of (3.4) in the form

$$w = \exp(iGs/2) \quad \ldots \quad (3.6)$$

and substituting in (3.5) we get a Ricatti equation of the type

$$H'+(iG)H+H^2+\rho\{(kD-M)-iGT+(iGD/2)\}-\rho^2D^2/4 = 0 \quad \ldots \quad (3.7a)$$

where

$$H = \phi'. \quad \ldots \quad (3.7b)$$

It may be noted that $w(s)$ of (3.4) is actually the solution of (3.2) for $\rho = 0$ and, therefore, the solution ϕ of (3.5) must be such that $\phi(s, o) = 0$. So also is $\phi'(s, o) = 0$ due to (2.8).

Now if we stipulate that a power series

$$H = \sum_{1}^{\infty} h_n\rho^n \quad \ldots \quad (3.8)$$

should satisfy (3.7) and that h_n are complex constants then we must have

$$h_n = \left[g_n+\sum_{j=1}^{n-1} h_jh_{n-j}\right](nk-iG)^{-1}, \quad n = 1, 2\ldots \quad \ldots \quad (3.9a)$$

where

$$\left.\begin{aligned} g_1 &= kD-M-iGT+(iGD/2) \\ g_2 &= -D^2/4 \\ g_n &= 0, \quad n > 2. \end{aligned}\right\} \quad \ldots \quad (3.9b)$$

and

The recursive relations (3.9) are obtained straightaway by substituting (3.8) in (3.7) and equating to zero the various coefficients of ρ^n.

If one assumes the sequence of complex constants $\{h_n\}$ is of bounded variation* and since $\sum_{n=1}^{\infty} \rho^n$ converges uniformly to $\rho/(1-\rho)$ in the closed interval $0 \leqslant \rho \leqslant \rho_0(\rho_0 < 1)$, it is not difficult to see that the series (3.8) must converge uniformly in the same interval and hence we must have, on integrating the series (3.8) term by term,

$$\phi = -\sum_{n=1}^{\infty} (h_n/nk)\rho^n. \qquad (3.10)$$

A particular solution of (3.2) is, therefore,

$$u_1 = \exp\left\{(iGs/2) - \sum_{n=1}^{\infty} (h_n/nk)\rho^n\right\}. \qquad (3.11)$$

Similarly another independent solution is

$$u_2 = \exp\left\{-(iGs/2) - \sum_{1}^{\infty} (\bar{h}_n/nk)\rho^n\right\} \qquad (3.12)$$

where $\bar{h}$ is the complex conjugate of h. Hence we have the following result.

If there exists a set of complex constants $h_n(n = 1, 2 \ldots)$ of bounded variation and given by (3.9) and $\bar{h}_n$ be their complex conjugates, then the complete solution of (2.1) is given by

$$\xi = A_1 \exp\left\{iGs + (\rho D/2k) - \sum_{1}^{\infty} (h_n/nk)\rho^n\right\}$$

$$+ A_2 \exp\left\{(\rho D/2k) - \sum_{1}^{\infty} (\bar{h}_n/nk)\rho^n\right\} \qquad (3.13)$$

where A_1, A_2 are arbitrary constants of integration. The perturbation series are uniformly convergent within the closed interval $0 \leqslant \rho \leqslant \rho_0(\rho_0 < 1)$. (3.13) is readily obtained by putting (3.11) and (3.12) for u in (3.1).

4. Oscillatory Stability and Boundedness of Limiting Motion

As $s \to \infty$, i.e. $\rho \to 0$, it could be seen from the analysis of Stone (1966) that for certain specific launching conditions

$$\xi(s) = (2/G)^{1/2} \exp\{i(G \mp G)s/2\}. \qquad (4.1)$$

However, from our present analysis it is obvious that all solutions of (2.1) are uniformly bounded, provided that conditions (3.9) are realized and that as $\rho \to 0$ we have

$$\xi(s) = A_1 \exp(iGs) + A_2. \qquad (4.2)$$

* $\{h_n\}$ is of bounded variation if $\sum_{n=1}^{\infty} |h_n - h_{n-1}| < \infty$, and therefore, $\sum_{n=1}^{\infty} (h_n - h_{n-1}) < \infty$ and since h_1 is finite, $\lim_{n\to\infty} h_n$ should necessarily exist which imply that the real and imaginary parts of (3.9) must separately converge.

Besides, from analytic theory of ordinary differential equations one can establish a few more (useful) sufficient conditions for boundedness of the limiting motion of the missile. For this we need the following results.

Theorem I—Sufficient conditions that all solutions of the differential equation

$$u''+\{1+q(x)\}u = 0 \quad (' = d/dx) \qquad (4.3)$$

be bounded are

$$(a) \int_{x_0}^{x} q(x_1)\, dx_1, \quad \int_{x_0}^{x} q(x_1) \sin 2x_1\, dx_1$$

and

$$\int_{x_0}^{x} q(x_1) \cos 2x_1\, dx_1$$

be uniformly bounded for

$$x_0 \leqslant x < \infty$$

and

$$(b) \int_{x_0}^{x} \left| q(x_1) \int_{x_1}^{x} \sin (x-x_1) \sin (x_1-x_2) q(x_2)\, dx_2 \right| dx_1 < \lambda_0 < 1,$$

for all $x \geqslant$ some fixed x_0.

Theorem II—Let $y(s)$ satisfy the differential equation

$$\{K(s)y'\}'+L(s) = 0 \quad (' = d/ds) \qquad (4.4)$$

where $K(s)$ and $L(s)$ are both positive functions of s and have a continuous derivative. Then the relative maxima of $|y|$ form an increasing or decreasing sequence according as $K(s)\,L(s)$ is decreasing or increasing.

Results embodied in Theorem II apply apparently to real solutions of (4.3) only. Proof of this theorem for $K(s) = 1$ was first given by Sonine and in the present form the result is due to Polya. For this, reference may be made to Szego (1950) (*see also* Hartman 1964).

Theorem I is a known result due to Bellman (1953), and also applies to boundedness of all real solutions of the differential equation.

Now in (3.2) if we put

$$s = (2/G)x \qquad (4.5)$$

the equation assumes the form (4.3)
where

$$q(x) = a \exp(-mx) - b \exp(-2mx) \qquad (4.6)$$

and

$$a = (4/G^2)\rho_0(kD-M), \; b = \rho_0 D^2/G^2, \; m = 2k/G. \qquad (4.7)$$

Due to a theorem of Winter (1949) it can be seen that all real solutions of (4.3) are oscillatory. If we apply the conditions of Theorem I above we shall precisely obtain a condition characterizing the boundedness of these oscillatory solutions of (4.3).

It may be seen straightaway due to (4.6) that all the integrals in (*a*) Theorem I are uniformly bounded in the interval

$$0 \leqslant x_0 < x < \infty$$

and the integral involved in condition (*b*) Theorem I can be written as

$$I(x_0, x) = \int_{x_0}^{x} | q(x_1) \sin (x-x_1) J(x_1, x) | \, dx_1 \quad .. \quad .. \quad (4.8)$$

where

$$\begin{aligned} J(x_1, x) &= \int_{x_1}^{x} \sin (x_1-x_2) q(x_2) \, dx_2 \\ &= a/(1+m^2) \exp(-mx) \{\cos (x_1-x) - m \sin (x_1-x)\} \\ &\quad -\{b/(1+4m^2)\} \exp(-2mx) \{\cos (x_1-x) - 2m \sin (x_1-x)\} \\ &\quad -\{a/(1+m^2)\} \exp(-mx_1) + \{b/(1+4m^2)\} \exp(-2mx_1) \quad .. \quad (4.9) \end{aligned}$$

and

q(x) is given by (4.6).

Now it may be seen that the sufficient conditions for boundedness of all (oscillatory) solutions of (4.3) imply that a positive x can be found such that,

$$I(x_0, x) < \lambda_0 < 1. \quad .. \quad .. \quad .. \quad .. \quad (4.10)$$

From (4.8) and (4.9) it is obvious that a sufficiently large positive x_0 can always be found such that (4.10) is realized. Thus we can say that the limiting oscillatory solutions are always bounded in the interval $0 \leqslant x_0 < x < \infty$ where x_0 is such that (4.10) is satisfied.

Since

$$\begin{aligned} I(x_0, x) &\leqslant \int_{x_0}^{x} | q(x_1) J(x_1 \, x) | \, dx_1 \\ &\leqslant \phi(x, x_0) - \phi(x, x) \quad .. \quad .. \quad .. \quad .. \quad (4.11) \end{aligned}$$

where

$$\begin{aligned} \phi(x, x_0) &= [(| a |/m) \exp(-mx_0) + (b/2m) \exp(-2mx_0)] \\ &\quad \times [\{| a | (1+m)/(1+m^2)\} \exp(-mx) + \{b(1+2m)/(1+4m^2)\} \\ &\quad \times \exp(-2mx)] + \{a^2/2m(1+m^2)\} \exp(-2mx_0) \\ &\quad + (| a | b/3m)\{(2+5m)/(1+m^2)(1+4m^2)\} \exp(-3mx_0) \\ &\quad + \{b^2/4m(1+4m^2)\} \exp(-4mx_0) \quad .. \quad .. \quad .. \quad .. \quad (4.12) \end{aligned}$$

a result which follows due to the fact that $\int | A+B+\ldots | \, dx \leqslant \int | A | \, dx + \int | B | \, dx + \ldots$ and $\left| \begin{matrix} \sin \theta \\ \cos \theta \end{matrix} \right| < 1$ and also since $\phi(x, x_0) > \phi(x, x)$ and min $\phi(x, x) = 0$, a sufficient condition for boundedness could now be written as

$$\phi(x, x_0) < \lambda_0 < 1 \quad .. \quad .. \quad .. \quad .. \quad (4.13)$$

and to look for an x_0 consistent with (4.13) we must have

$$\phi(x_0, x_0) < \lambda_0 < 1 \quad .. \quad .. \quad .. \quad .. \quad (4.14)$$

as

$$\phi(x_0, x_0) \geqslant \phi(x, x_0).$$

The lower bound of x, consistent with the inequality (4.14) cannot, however, be less than the only positive root of the equation

$$\phi(x_0, x_0) = 1,$$

i.e.

$$\{a^2(3+2m)/2m(1+m^2)\} \exp(-2mx_0) + \{|a| b(24m^3+28m^2+15m+13)/6m(1+m^2)(1+4m^2)\} \exp(-3mx_0)+\{b^2(3+4m)/4m(1+4m^2)\} \exp(-4mx_0)-1 = 0. \quad (4.15)$$

Corresponding to this positive x_0 thus found out, i.e. after the rocket passes through the corresponding point $(Gs_0/2)$ in its trajectory, the subsequent oscillatory motion should be bounded as can be seen from (3.1) and (4.5). This is more strongly so if $D > 0$.

Coming to Theorem II we may reduce (2.1) to (4.4) through the substitution

$$\xi = y \exp(iGs/2) \quad .. \quad .. \quad .. \quad .. \quad (4.16)$$

where

$$\left.\begin{aligned} K(s) &= \exp(-D\rho/k) \\ L(s) &= \exp(-D\rho/k)\{(G^2/4)-\rho M\} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (4.17)$$

with the restriction

$$D = 2T. \quad .. \quad .. \quad .. \quad .. \quad (4.18)$$

In the present case it may be noted that

$$K(s) > 0 \quad \text{and} \quad L(s) > 0 \text{ if } (G^2/4)-\rho M > 0. \quad .. \quad .. \quad (4.19)$$

The amplitudes of the oscillations should increase or decrease with increasing s according as

$$d\{L(s)K(s)\}/ds \lesseqgtr 0,$$

i.e. according as

$$kM+(G^2D/2)-2\rho MD \lesseqgtr 0. \quad .. \quad .. \quad .. \quad (4.20)$$

Hence we have the following scheme determining the decay or growth of oscillations of the missile exiting the atmosphere.

Static stability parameter	Dynamic stability parameter	Amplitude characteristics
$M < 0$	$D < 0$	amplitudes grow at all heights.
$M < 0$	$D > 0$	amplitudes grow or decay for $\rho \gtrless \rho_c$.
$M > 0$	$D > 0$	amplitudes decay for $\rho < (G^2/4M)$ and therefore for $\rho < \rho_c$.
$M > 0$	$D < 0$	amplitudes decay for $\rho < \rho_c$ and grow for $(G^2/4M) > \rho > \rho_c$.

In the scheme shown above it may be noted that

$$\rho_c = (k/2D+G^2/4M). \quad .. \quad .. \quad .. \quad .. \quad (4.21)$$

These conditions are naturally more general than the corresponding ones given by Stone (1966) and clearly apply to rolling missiles also having aerodynamic restrictions of the type (4.18). The critical density as estimated from (4.21) for the rocket [cf. Stone (1966)] with the aerodynamic characteristics,

$$D = 46{\cdot}722\times10^{-5}$$
$$M = 1{\cdot}835\times10^{-2}$$
$$G = 2{\cdot}012\times10^{-2}$$
$$\rho_0 = {\cdot}00238$$
$$\beta = 1/22000$$

and

$$k = 4{\cdot}54\times10^{-5}$$

is now seen to be $4{\cdot}86\times10^{-6}$ corresponding to a height of 41·5 km.

Concluding Remarks

The foregoing study shows that valuable information regarding the stability of oscillatory motion of a coasting rocket exiting the atmosphere could be obtained from a study of the general asymptotic behaviour of the dynamical equations. For the purpose it is not so much necessary to resort to approximate solutions where the information could at the most be partial.

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