THERMOELASTIC VIBRATIONS OF A SIMPLY SUPPORTED RECTANGULAR PLATE PRODUCED BY TEMPERATURES PRESCRIBED ON THE FACES

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The present paper is concerned with the determination of dynamic thermal deflection of a simply supported rectangular plate of thickness h of which one face is at the zero temperature while the other face has prescribed temperature. The solution is obtained in the form of a double trigonometric series.

INTRODUCTION

It has been pointed out by Boley and Barber (1957) that the analysis of the behaviour of various types of structural elements subjected to elevated temperatures is often of paramount importance in the design of high-speed aircraft or projectiles and that in order to perform a practical analysis of this kind, it is necessary to examine the importance of the role of inertia. In fact, the assumption that the inertia may be disregarded does not seem reasonable for structural elements exposed to rapid heating, as may be encountered, for example, in high-speed propulsion units or in problems of thermal shock. A thermoelastic analysis in which inertia term is retained will be referred to as a 'dynamic solution'. The authors mentioned above have discussed thermal vibration of a simply supported rectangular plate under a uniform step heat input over one face with the other face thermally insulated. They have assumed that the temperature is propagated only along the thickness, which is, in fact, valid for a very thin plate. In the present paper a more realistic problem has been considered. The temperature is supposed to satisfy the general quasi-static heat conduction equation in the plate and thermal vibration of simply supported rectangular plate of thickness subjected to a non-uniform temperature distribution on one face with zero temperature on the other has been discussed. Results of two other cases corresponding to the constant temperature distribution and temperature prescribed at a point on one face of the plate have been briefly presented.

Formulation of the Problem

Governing Equations

Let us consider a free simply supported rectangular plate occupying the space

$$0 \leqq x \leqq a, \quad 0 \leqq y \leqq b, \quad -\frac{h}{2} \leqq z \leqq \frac{h}{2}.$$

We assume that the temperature T is given by

$$T(x, y, z, t) = \left(z+\frac{h}{2}\right) \sum_{m} \sum_{n} f_{mn}(t) \,.\, \sin \alpha_m x \sin \beta_n y$$

where

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b}. \quad \cdots \quad (1)$$

$f_{mn}(t)$ will be determined from the heat conduction equation. In this case, it is found that the temperature on the face $z = -h/2$ is zero.

As a particular example, we consider the case

$$[T]_{\substack{z=\frac{h}{2}\\ t=0}} = T_0 x(a-x) \,.\, y \,.\, (b-y) \quad \cdots \quad (2)$$

T_0, being a constant.

This expression satisfies the condition that the temperature is zero at the edges of the surface $z = h/2$.

Heat conduction equation being

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \cdot \frac{\partial T}{\partial t}.$$

We have

$$\frac{d f_{mn}(t)}{dt} = -k({\alpha_m}^2+{\beta_n}^2) \,.\, f_{mn}(t).$$

Hence

$$f_{mn}(t) = \mu_{mn} \,.\, e^{-k\left(\alpha_m^2+\beta_n^2\right) \,.\, t}$$

μ_{mn} being a constant and k is the thermal diffusivity. The constant μ_{mn} can be found from the nature of the temperature prescribed on the upper face.

Now initially,

$$T_0 x(a-x) \,.\, y(b-y) = h \sum_m \sum_n \mu_{mn} \,.\, \sin \alpha_m x \,.\, \sin \beta_n y.$$

By Fourier's method, we get

$$\mu_{mn} = \frac{64 T_0 a^2 b^2}{\pi^6 \,.\, m^3 n^3 \,.\, h}; \quad m, n = 1, 3, 5, \ldots$$

Hence the temperature distribution is given by

$$T(x, y, z, t) = \left(z+\frac{h}{2}\right) \cdot \frac{64 T_0 a^2 b^2}{\pi^6 \,.\, h} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin \alpha_m x \,.\, \sin \beta_n y}{m^3 n^3} \,.\, e^{-\gamma_{mn} \,.\, t}$$

where

$$\gamma_{mn} = k(\alpha_m{}^2+\beta_n{}^2).$$

The above result gives

$$M_T = \alpha E \int_{-\frac{h}{2}}^{\frac{h}{2}} zT\, dz$$

$$= \lambda_0 \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin \alpha_m x \,.\, \sin \beta_n y}{m^3 n^3} \,.\, e^{-\gamma_{mn}\,.\,t} \quad \ldots \quad \ldots \quad (3)$$

where

$$\lambda_0 = \frac{16}{3} \,.\, \frac{T_0 a^2 b^2 h^2 \alpha E}{\pi^6};$$

α, E denoting coefficient of linear thermal expansion and Young's modulus of the material of the plate respectively.

Differential equation for normal deflection $w(x, y, t)$ of the plate is (cf. Boley and Weiner 1960, p. 384, 407).

$$D\nabla_1{}^4 w+\rho h \frac{\partial^2 w}{\partial t^2} = -\frac{\nabla_1{}^2 M_T}{(1-\nu)} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$D = \dfrac{Eh^3}{12(1-\nu^2)}$ denoting bending rigidity of the plate and ρ the density of the material of the plate. The constant ν denotes Poisson's ratio of the plate.

Initial and Boundary Conditions

(i) Initial conditions:

$$\left.\begin{aligned} w = \frac{\partial w}{\partial t} = 0, \quad t = 0 \\ \text{for} \quad 0 \leqq x \leqq a, \quad 0 \leqq y \leqq b \end{aligned}\right\} \,. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

(ii) Boundary condition: The plate being simply supported,

$$\left.\begin{aligned} & w(0, y) = w(a, y) = 0, \quad \text{for all } y \text{ in } 0 \leqq y \leqq b \\ & w(x, 0) = w(x, b) = 0, \quad \text{for all } x \text{ in } 0 \leqq x \leqq a \\ & \frac{\partial^2 w}{\partial x^2}+\frac{M_T}{(1-\nu)D} = 0, \quad \text{at } x = 0, a \\ & \frac{\partial^2 w}{\partial y^2}+\frac{M_T}{(1-\nu)D} = 0, \quad \text{at } y = 0, b \end{aligned}\right\} \,. \quad \ldots \quad (6)$$

Solution of the Problem

We assume

$$w(x, y, t) = \sum_{m=1,3,5\ldots}^{\infty} \sum_{n=1,3,5\ldots}^{\infty} \phi_{mn}(t) \,.\, \sin \alpha_m x \,.\, \sin \beta_n y. \quad \ldots \quad (7)$$

Inserting the above result in (4), we deduce the differential equation satisfied by $\phi_{mn}(t)$ as

$$\frac{d^2\phi_{mn}(t)}{dt^2}+\lambda_{mn}^2\phi_{mn}(t) = \frac{\lambda_0(\alpha_m^2+\beta_n^2) . e^{-\gamma_{mn} . t}}{\rho h(1-\nu) . m^3n^3} \quad .. \quad .. \quad (8)$$

with

$$\phi_{mn}(0) = \dot{\phi}_{mn}(0) = 0. \quad .. \quad .. \quad .. \quad (9)$$

Solving the equation by the use of Laplace transforms,

$$\phi_{mn}(t) = \frac{\lambda_0(\alpha_m^2+\beta_n^2)}{\rho h(1-\nu) . m^3n^3}\left[\frac{\gamma_{mn} . \sin(\lambda_{mn} . t)}{\lambda_{mn}(\lambda_{mn}^2+\gamma_{mn}^2)} - \frac{\cos(\lambda_{mn} . t)}{(\lambda_{mn}^2+\gamma_{mn}^2)}+\frac{e^{-\gamma_{mn} . t}}{(\lambda_{mn}^2+\gamma_{mn}^2)}\right].$$

Hence the thermal deflection w is given by

$$\begin{aligned} w(x, y, t) = \frac{\lambda_0}{\rho h(1-\nu)} . \sum_{m=1, 3, 5 \ldots}^{\infty} \sum_{n=1, 3, 5 \ldots}^{\infty} & \frac{(\alpha_m^2+\beta_n^2)}{m^3n^3\lambda_{mn}(\lambda_{mn}^2+\gamma_{mn}^2)} \\ & \times\{\gamma_{mn} . \sin(\lambda_{mn} . t)-\lambda_{mn} . \cos(\lambda_{mn} . t) \\ & +\lambda_{mn} . e^{-\gamma_{mn} . t}\} . \sin \alpha_m x . \sin \beta_n y \quad .. \quad .. \quad .. \quad (10) \end{aligned}$$

where

$$\lambda_{mn}^2 = \frac{D}{\rho h}(\alpha_m^2+\beta_n^2)^2$$

$$\gamma_{mn} = k(\alpha_m^2+\beta_n^2).$$

It is found from (3) and (10) that the boundary conditions (6) are obviously satisfied.

Discussion

(i) The expression for normal deflection shows that after a long time there is a vibration with period $\frac{2\pi}{\lambda_{mn}}$

where

$$\lambda_{mn} = \sqrt{\frac{D}{\rho h}} . \pi^2 . \left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right).$$

This result is independent of temperature which only affects the amplitude.

The lowest frequency μ is given by

$$\mu = \frac{1}{2}\sqrt{\frac{D}{\rho h}} . \pi\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

(ii) Dropping the inertia term, we get the static solution from the equation

$$D\nabla_1^4 w+\frac{\nabla_1^2 M_T}{(1-\nu)} = 0 \quad .. \quad .. \quad .. \quad .. \quad (11)$$

under the boundary conditions (6).

The solution is given by

$$w_s(x, y, t) = \frac{\lambda_0}{D(1-\nu)} \sum_{m=1, 3, \ldots}^{\infty} \sum_{n=1, 3, \ldots}^{\infty} \frac{\sin \alpha_m x . \sin \beta_n y . e^{-\gamma_{mn} . t}}{m^3n^3(\alpha_m^2+\beta_n^2)}. \quad (12)$$

Additional Results

The method can be applied to the other cases also. For example, we consider the following cases:

(i) $[T]_{\substack{z=\frac{h}{2}\\ t=0}} = T_0 = \text{constant}, \quad \text{for } 0 \leqq x \leqq a, \quad 0 \leqq y \leqq b$

with

$$[T]_{\substack{x=0,\,a\\ y=0,\,b}} = 0, \quad \text{for } t > 0.$$

The results are given by

$$T(x,y,z,t) = \left(z+\frac{h}{2}\right)\cdot\frac{16T_0}{h\pi^2}\sum_{m=1,3\ldots}^{\infty}\;\sum_{n=1,3\ldots}^{\infty}\frac{\sin\alpha_m x\,.\,\sin\beta_n y}{mn}\,.\,e^{-\gamma_{mn}\,.\,t}$$

$$M_T = \lambda_0\sum_{m=1,3\ldots}^{\infty}\;\sum_{n=1,3\ldots}^{\infty}\frac{\sin\alpha_m x\,.\,\sin\beta_n y}{mn}\,.\,e^{-\gamma_{mn}\,.\,t}$$

where

$$\lambda_0 = \frac{4}{3}\cdot\frac{\alpha E h^2 T_0}{\pi^2}.$$

Hence

$$w(x,y,t) = \frac{\lambda_0}{\rho h(1-\nu)}\sum_{m=1,3\ldots}^{\infty}\;\sum_{n=1,3\ldots}^{\infty}\frac{(\alpha_m^2+\beta_n^2)}{mn\,.\,\lambda_{mn}(\lambda_{mn}^2+\gamma_{mn}^2)}$$
$$\times\{\gamma_{mn}\,.\,\sin(\lambda_{mn}\,.\,t)-\lambda_{mn}\,.\,\cos(\lambda_{mn}\,.\,t)$$
$$+\lambda_{mn}\,.\,e^{-\gamma_{mn}\,.\,t}\}\,.\,\sin\alpha_m x\,.\,\sin\beta_n y.$$

(ii) $[T]_{\substack{z=\frac{h}{2}\\ t=0}} = T_0\delta(x-\xi)\delta(y-\eta)$

where T_0 is a constant and $\delta(x)$ denotes Dirac's delta function.

Also

$$0 < \xi < a, \quad 0 < \eta < b.$$

The final results are given by

$$T(x,y,z,t) = \left(z+\frac{h}{2}\right)\cdot\frac{4T_0}{abh}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\alpha_m\xi\,.\,\sin\beta_n\,.\,\eta\,.\,\sin\alpha_m x\,.\,\sin\beta_n y\,.\,e^{-\gamma_{mn}\,.\,t}$$

$$M_T = \lambda_0\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\alpha_m\xi\,.\,\sin\beta_n\,.\,\eta\,.\,\sin\alpha_m x\,.\,\sin\beta_n y\,.\,e^{-\gamma_{mn}\,.\,t}$$

where

$$\lambda_0 = \frac{\alpha E T_0\,.\,h^2}{3ab}.$$

Hence

$$w(x, y, t) = \frac{\lambda_0}{\rho h(1-\nu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\alpha_m^2+\beta_n^2) \, . \sin \alpha_m \xi \, . \sin \beta_n \, . \eta}{\lambda_{mn} \, . \, (\lambda_{mn}^2+\gamma_{mn}^2)}$$

$$\times \{\gamma_{mn} \, . \sin (\lambda_{mn} \, . \, t) - \lambda_{mn} \, . \cos (\lambda_{mn} \, . \, t)$$

$$+ \lambda_{mn} \, . \, e^{-\gamma_{mn} \, . \, t}\} \, . \sin \alpha_m x \, . \sin \beta_n y$$

where

$$\lambda_{mn}^2 = \frac{D}{\rho h}(\alpha_m^2+\beta_n^2)^2,$$

$$\gamma_{mn} = k(\alpha_m{}^2+\beta_n{}^2).$$

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