ON K_λ-CURVATURE OF A VECTOR FIELD IN FINSLER SPACE

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The K_λ curves of a Riemannian space have been initially studied by Tsagas (1969). The K_λ curves of a subspace of a Finsler space have been studied by Agrawal (1970). The union curvature of a vector field in Finsler space was studied by Nisha Rani (1968). The object of the present paper is to study K_λ-curvature of a vector field.

1. FUNDAMENTAL FORMULAE

In the present sections we shall outline some basic concepts and fundamental formulae from the theory of Finsler subspace as given by Eliopoulos (1959).

Let an m-dimensional subspace F_m given by equations $x^i = x^i(u^\alpha)$ $(i = 1, 2, \ldots, n;\ \alpha = 1, 2, \ldots, m)$ be immersed in an n-dimensional Finsler space F_n. Consider a curve C: $x^i = x^i(s)$ of F_m where s denotes the arc length. The components x'^i and u'^α of the unit tangent vector of C are given by $x'^i = X^i_\alpha u'^\alpha$ where $X^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$. The metric tensors $g_{ij}(x, x')$, $g_{\alpha\beta}(u, u')$ of F_n and F_m are such that

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x')X^i_\alpha X^j_\beta. \qquad .. \quad .. \quad .. \quad (1.1)$$

The vectors $n^i_{(\nu)}$ $(\nu = m+1, \ldots, n)$ and $n^{*i}_{(\nu)}(u, u')$ $(\nu = m+1, \ldots, n)$ satisfying the conditions

$$g_{ij}(x, n_{(\nu)})X^i_\alpha n^j_{(\nu)} = n_{(\nu)i}X^i_\alpha = 0, \quad g_{ij}(x, n_{(\nu)})n^i_{(\nu)}n^j_{(\nu)} = 1 \qquad .. \quad (1.2)$$

and

$$\left.\begin{aligned} &g_{ij}(x, x')X^i_\alpha n^{*j}_{(\nu)} = 0, \quad g_{ij}(x, n^*_{(\nu)})n^{*i}_{(\nu)}n^{*j}_{(\nu)} = 1 \\ &g_{ij}(x, x')n^{*i}_{(\nu)}n^{*j}_{(\mu)} = \delta^{(\nu)}_{(\mu)}\psi_{(\nu)}(x, x') \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (1.3)$$

(no summation w.r.t. ν)

are called (respectively) the normal vectors and the secondary normal vectors of the subspace. The induced connection parameter $\Gamma^{*\epsilon}_{\alpha\beta}(u, u')$ of F_m is given by

$$\Gamma^{*\epsilon}_{\alpha\beta} = X^\epsilon_i\left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^{*i}_{hk}X^h_\alpha X^k_\beta\right)$$

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where $X_i^\epsilon = g^{\epsilon\alpha}(u, u')g_{ij}(x, x')X_\alpha^j$ and Γ^{*i}_{hk} are the connection coefficients of the embedding space F_n.

The parameter $\Gamma^{*\epsilon}_{\alpha\beta}$ is used in the definition of what have been called 'the tensor (or mixed) derivative' and 'the induced covariant derivative' in the subspace. In particular the tensor derivative X^i_α is given by

$$X^i_{\alpha\beta} = X^i_{\alpha,\beta} = (\partial^2 x^i/\partial u^\alpha \partial u^\beta) - X^i_\gamma \Gamma^{*\gamma}_{\alpha\beta} + \Gamma^{*i}_{hk} X^h_\alpha X^k_\beta$$

whence $\Omega_{(\nu)\alpha\beta} = n_{(\nu)i} X^i_{\alpha\beta}$. The $(n-m)$ tensors with the components $\Omega_{(\nu)\alpha\beta}$ are called the second fundamental tensors of the subspace. It is further shown that

$$X^i_{\alpha\beta} = \sum_\nu \Omega^*_{(\nu)\alpha\beta} n^{*i}_{(\nu)} \quad .. \quad .. \quad .. \quad .. \quad (1.4)$$

where $\Omega^*_{(\nu)\alpha\beta}(u, u')$ are the components of the secondary second fundamental tensors of F_m. The following results given by Eliopoulos (1959) will be used in the sequel. We have

$$X^i_{\alpha\beta} = X^i_\gamma \sum_\nu A_{(\nu)\alpha\beta} M^\gamma_{(\nu)} + \sum_\nu A_{(\nu)\alpha\beta} n^i_{(\nu)} \quad .. \quad .. \quad (1.5)$$

where $A_{(\nu)\alpha\beta}$ and $M^\alpha_{(\nu)}$ satisfy

$$\Omega_{(\nu)\alpha\beta} = \sum_\nu A_{(\nu)\alpha\beta} \cos(n_{(\mu)}, n_{(\nu)}) \quad \text{and} \quad M^\gamma_{(\nu)} = -n^i_{(\nu)} X^\gamma_i .$$

Let v be a vector field of the subspace whose components v^α in F_m are such that $v^i = x^i_\alpha v^\alpha$. The components $\delta v^i/\delta s$ of the derived vector (with respect to F_n) of v^i in the direction of a curve $C: x^i = x^i(s)$ (of F_m) are given by

$$\frac{\delta v^i}{\delta s} = \frac{\delta v^\alpha}{\delta s} X^i_\alpha + X^i_{\alpha\beta} v^\alpha \frac{du^\beta}{ds} \quad .. \quad .. \quad .. \quad .. \quad (1.6)$$

where $(\delta v^\alpha/\delta s)$ are components of the associate curvature vector (induced) of the vector field.

If $(\delta v^\alpha/\delta s) = 0$, the vector field is said to be parallel (along C) in F_m.

2. K_λ Curves

Consider a congruence of curves given by vector $\lambda^i(u)$ such that through each point of F_m there passes one curve of the congruence. The vector field λ can be expressed linearly in terms of X^i_α and $n^i_{(\nu)}$, i.e.

$$\lambda^i(u) = t^\alpha X^i_\alpha + \sum_\nu C_{(\nu)} n^i_{(\nu)} \quad .. \quad .. \quad .. \quad (2.1)$$

where t^α are contravariant components of a vector in F_m and $C_{(\nu)}$ are scalars. This is normalized by the condition

$$g_{ij}(x, x')\lambda^i\lambda^j = 1. \quad .. \quad .. \quad .. \quad .. \quad (2.2)$$

Relation (2.2) in view of (2.1), (1.1) and (1.2) takes the form†

$$g_{\alpha\beta}(u,\ u')t^{\alpha}t^{\beta}=1-2\sum_{\nu}C_{(\nu)}n_{(\nu)i}t^{\alpha}X^{i}_{\alpha}-\sum_{\mu,\ \nu}C_{(\mu)}C_{(\nu)}g_{ij}n^{i}_{(\mu)}n^{j}_{(\nu)}. \quad .. \quad (2.3)$$

Definition 2.1—The curve C is said to be a K_{λ} (or secondary K_{λ}) curve relative to the vector field v if the variety determined by the two associate curvature vectors of v contains the congruence λ (or λ^{*}).

K_{λ} curve relative to v is given by

$$\lambda^{i}=u\frac{\delta v^{\alpha}}{\delta s}X^{i}_{\alpha}+w\left(\frac{\delta v^{\alpha}}{\delta s}X^{i}_{\alpha}+X^{i}_{\alpha\beta}v^{\alpha}\frac{du^{\beta}}{ds}\right) \quad .. \quad .. \quad .. \quad (2.4)$$

where u and w are parameters to be determined. The above equation in view of (2.1) and (1.5) takes the form

$$t^{\alpha}X^{i}_{\alpha}+\sum_{\nu}C_{(\nu)}n^{i}_{(\nu)}=X^{i}_{\alpha}\left[(u+w)\frac{\delta v^{\alpha}}{\delta s}+w\sum_{\nu}A_{(\nu)\beta\gamma}M^{\alpha}_{(\nu)}v^{\beta}\frac{du^{\gamma}}{ds}\right]+w\left(\sum_{\nu}A_{(\nu)\alpha\beta}n^{i}_{(\nu)}v^{\alpha}\frac{du^{\beta}}{ds}\right). \quad .. \quad (2.5)$$

Since the vectors X^{i}_{α} and $n^{i}_{(\nu)}$ are linearly independent, we can write

$$t^{\alpha}=(u+w)\frac{\delta v^{\alpha}}{\delta s}+w\sum_{\nu}A_{(\nu)\beta\gamma}M^{\alpha}_{(\nu)}v^{\beta}\frac{du^{\gamma}}{ds} \quad .. \quad .. \quad (2.6)$$

$$C_{(\nu)}=wA_{(\nu)\alpha\beta}v^{\alpha}\frac{du^{\beta}}{ds}. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.6a)$$

Multiplying (2.6) by $g_{\alpha\beta}(u,\ u')t^{\beta}$ and summing with respect to β, we have

$$1-2\sum_{\nu}C_{(\nu)}n_{(\nu)i}t^{\alpha}X^{i}_{\alpha}-\sum_{\mu,\ \nu}C_{(\nu)}C_{(\mu)}g_{ij}(x,\ x')n^{i}_{(\nu)}n^{j}_{(\mu)}$$
$$=(u+w)\frac{\delta v^{\alpha}}{\delta s}g_{\alpha\beta}t^{\beta}+w\sum_{\nu}g_{\alpha\beta}M^{\alpha}_{(\nu)}t^{\beta}A_{(\nu)\delta\gamma}v^{\delta}\frac{du^{\gamma}}{ds}. \quad .. \quad (2.7)$$

Eliminating u, w from (2.6) we get

$$\left[1-2\sum_{\nu}C_{(\nu)}n_{(\nu)i}t^{\gamma}X^{i}_{\gamma}-\sum_{\mu,\ \nu}C_{(\mu)}C_{(\nu)}g_{ij}n^{i}_{(\nu)}n^{j}_{(\mu)}-\frac{C_{(\nu)}}{A_{(\nu)\beta\gamma}v^{\beta}\dfrac{du^{\gamma}}{ds}}\sum_{\nu}A_{(\nu)\delta\gamma}v^{\delta}\frac{du^{\gamma}}{ds}g_{\epsilon\beta}M^{\epsilon}_{(\nu)}t^{\beta}\right]\frac{\delta v^{\alpha}}{\delta s}$$
$$-\left[t^{\alpha}-\frac{C_{(\nu)}}{A_{(\nu)\gamma\beta}v^{\beta}\dfrac{du^{\gamma}}{ds}}\sum_{\nu}A_{(\nu)\delta\epsilon}v^{\delta}\frac{du^{\epsilon}}{ds}M^{\alpha}_{(\nu)}\right]g_{\gamma\beta}\frac{\delta v^{\gamma}}{\delta s}t^{\beta}=0 \quad .. \quad (2.8)$$

where

$$C_{(\nu)}/A_{(\nu)\gamma\beta}v^{\gamma}\frac{du^{\beta}}{ds} \text{ is independent of } \nu. \quad .. \quad .. \quad .. \quad (2.8a)$$

† We have denoted $n_{(\nu)i}=g_{ij}(x,\ x')n^{j}_{(\nu)}$ for brevity. It may be noted that $n_{(\nu)i}X^{i}_{\alpha}\neq 0$.

Theorem 2.1—The differential equation (2.8) subject to the condition (2.8a) represents the K_λ curve of the vector field with respect to the congruence λ.

For a hypersurface F_{n-1} the congruence λ is given by

$$\lambda^i = t^\alpha X^i_\alpha + En^i \qquad .. \quad .. \quad .. \quad .. \quad (2.9)$$

where E is a scalar.

The differential equation for the K_λ curve of the vector field takes the form

$$g_{\alpha\beta}\frac{\delta v^\alpha}{\delta s}t^\beta(t^\gamma - EM^\gamma) + (2En_i t^i + Eg_{\alpha\beta}M^\alpha t^\beta)\frac{\delta v^\gamma}{\delta s} + (E^2 g_{ij}n^i n^j - 1)\frac{\delta v^\gamma}{\delta s} = 0. \qquad (2.10)$$

A vector with the components

$${}_1\eta^\gamma_v = (t^\gamma - EM^\gamma)g_{\alpha\beta}\frac{\delta v^\alpha}{\delta s}t^\beta + E(2n_i t^i + g_{\alpha\beta}M^\alpha t^\beta)\frac{\delta v^\gamma}{\delta s} + (E^2 g_{ij}n^i n^j - 1)\frac{\delta v^\gamma}{\delta s} \qquad (2.11)$$

will be called the K_λ curvature vector of the vector field. The magnitude ${}_vK_\lambda$ of this vector will be called the K_λ curvature of v. We have, therefore,

$$\begin{aligned} {}_vK^2_\lambda = g_{\gamma\delta}\Big\{ & (t^\gamma - EM^\gamma)g_{\alpha\beta}\frac{\delta v^\alpha}{\delta s}t^\beta + E(2n_i t^i + g_{\alpha\beta}M^\alpha t^\beta)\frac{\delta v^\gamma}{\delta s} \\ & + (E^2 g_{ij}n^i n^j - 1)\frac{\delta v^\gamma}{\delta s}\Big\}\Big\{(t^\delta - EM^\delta)g_{\epsilon\theta}\frac{\delta v^\epsilon}{\delta s}\cdot t^\theta \\ & + E(2n_j t^j + g_{\alpha\beta}M^\alpha t^\beta)\frac{\delta v^\delta}{\delta s} + (E^2 g_{ij}n^i n^j - 1)\frac{\delta v^\delta}{\delta s}\Big\}. \qquad .. \quad (2.12) \end{aligned}$$

Theorem 2.2—A necessary and sufficient condition that a curve be a K_λ curve relative to v is that the K_λ curvature vector vanishes identically.

Theorem 2.3—A sufficient condition that the K_λ curvature of the vector field be proportional to the associate curvature of the vector field (with respect to F_{n-1}) is that the congruence λ is a normal congruence.

PROOF: If the congruence is normal, $t^\alpha = 0$, and the theorem follows immediately.

Particular Cases

(i) Suppose that the vector field is tangential to the curve C. The equation of the K_λ curve of F_{n-1} reduces to

$$g_{\alpha\beta}p^\alpha t^\beta(t^\gamma - EM^\gamma) + E(2n_i t^i + g_{\alpha\beta}M^\alpha t^\beta)p^\gamma + (E^2 g_{ij}n^i n^j - 1)p^\gamma = 0 \qquad .. \quad (2.13)$$

(ii) The vector field v under consideration is a parallel vector field of the hypersurface. Under this condition the equation (2.10) is identically satisfied. Hence, we have the following theorem:

Theorem 2.4—If a vector field undergoes a parallel displacement along a curve C, then C is a K_λ curve of the vector field relative to any congruence.

3. Secondary K_λ Curve

In this section we shall consider a congruence λ^* which is a function of a line element (u, u'). The vector field λ^* can be expressed linearly in terms of X^i_α and $n^{*i}_{(\nu)}$ as

$$\lambda^{*i} = t^{*\alpha}X^i_\alpha + \sum_\nu C^*_{(\nu)} n^{*i}_{(\nu)} \quad .. \quad .. \quad .. \quad (3.1)$$

where $C^*_{(\nu)}$ is a scalar. This vector is normalized by the condition

$$g_{ij}(x, x')\lambda^{*i}\lambda^{*j} = 1. \quad .. \quad .. \quad .. \quad .. \quad (3.2)$$

The secondary K_λ curve of the vector field is defined in the preceding section. Assuming that C is a secondary K_λ curve relative to the vector field v, we have

$$\lambda^{*i} = u^* \frac{\delta v^\alpha}{\delta s} X^i_\alpha + w^* \frac{\delta v^i}{\delta s} \quad .. \quad .. \quad .. \quad .. \quad (3.3)$$

where u^* and w^* are parameters to be determined.
Equation (3.3) in view of (3.1), (1.4) and (1.6) can be written as

$$t^{*\alpha}X^i_\alpha + \sum_\nu C^*_{(\nu)} n^{*i}_{(\nu)} = (u^*+w^*)\frac{\delta v^\alpha}{\delta s} X^i_\alpha + w^* \sum_\nu \Omega^*_{(\nu)\alpha\beta} v^\alpha \frac{du^\beta}{ds} n^{*i}. \quad .. \quad (3.4)$$

Since X^i_α and $n^{*i}_{(\nu)}$ are linearly independent, we have

$$t^{*\alpha} = (u^*+w^*)\frac{\delta v^\alpha}{\delta s} \quad .. \quad .. \quad .. \quad .. \quad (3.5)$$

$$C^*_{(\nu)} = w^* \Omega^*_{(\nu)\alpha\beta} v^\alpha \frac{du^\beta}{ds}. \quad .. \quad .. \quad .. \quad .. \quad (3.5a)$$

Multiplication of (3.5) by $g_{\alpha\beta}t^{*\beta}$ and summation with respect to β give

$$(u^*+w^*)\left(g_{\alpha\beta}\frac{\delta v^\alpha}{\delta s} t^{*\beta}\right) = \left(1 - \sum_\mu (C^*_{(\mu)})^2 \psi_{(\mu)}\right). \quad .. \quad .. \quad (3.6)$$

Eliminating u^*, w^* from (3.6) and (3.5), we obtain

$$\frac{\delta v^\alpha}{\delta s} - \left(g_{\beta\gamma}\frac{\delta v^\beta}{\delta s} t^{*\gamma}\right)\left(1 - \sum_\mu C^{*2}_{(\mu)}\psi_{(\mu)}\right)^{-1} t^{*\alpha} = 0. \quad .. \quad .. \quad (3.7)$$

Hence we have the following theorem:

Theorem 3.1—The differential equation (3.7) represents a secondary K_λ curve of the vector field relative to the (non-normal) congruence λ^*.

For a hypersurface F_{n-1} the congruence λ^* is given by

$$\lambda^{*i} = t^{*\alpha}X^i_\alpha + C^* n^{*i}. \quad .. \quad .. \quad .. \quad .. \quad (3.8)$$

The equations for the secondary K_λ curve of a vector field relative to the congruence λ^* are given by

$$\frac{\delta v^\alpha}{\delta s} - g_{\beta\gamma}\frac{\delta v^\beta}{\delta s} t^{*\gamma}(1 - C^{*2}\psi)^{-1} t^{*\alpha} = 0. \quad .. \quad .. \quad .. \quad (3.9)$$

The quantities

$$ {}_1\eta_v^{*\alpha} = \frac{\delta v^\alpha}{\delta s} - g_{\beta\gamma}\frac{\delta v^\beta}{\delta s} t^{*\gamma}(1-C^{*2}\psi)^{-1}t^{*\alpha} \quad .. \quad .. \quad .. \quad (3.10) $$

are the contravariant components of a vector in F_{n-1}. This vector is called the secondary K_λ curvature vector of the vector field in F_{n-1}. We have, therefore,

$$ {}_vK_\lambda^{*2} = g_{\alpha\beta}\left(\frac{\delta v^\alpha}{\delta s} - g_{\delta\gamma}\frac{\delta v^\delta}{\delta s} t^{*\gamma}(1-C^{*2}\psi)^{-1}t^{*\alpha}\right)\left(\frac{\delta v^\beta}{\delta s} - t^{*\beta}g_{\epsilon\theta}\frac{\delta v^\epsilon}{\delta s} t^{*\theta}(1-C^{*2}\psi)^{-1}\right). $$

.. (3.11)

We can now state the following theorems:

Theorem 3.2—A necessary and sufficient condition that a curve be a secondary K_λ curve of a vector field is that the K_λ curvature of the vector field vanishes identically.

Theorem 3.3—A necessary and sufficient condition that the secondary K_λ curvature of the vector field be equal to the associate curvature of the vector field (with respect to F_{n-1}) is that the congruence $\lambda^*(u, u')$ is a normal congruence.

PROOF: Let the congruence λ^* be a normal congruence, i.e. $t^{*\alpha} = 0$, hence

$$ {}_vK_\lambda^{*2} = g_{\alpha\beta}\frac{\delta v^\alpha}{\delta s}\frac{\delta v^\beta}{\delta s} = {}_vK_a^2 $$

where ${}_vK_a$ is the associate curvature of the vector field with respect to the hypersurface.

Conversely if ${}_vK_\lambda^* = {}_vK_a$, then $t^{*\alpha} = 0$.

Hence the proposition.

Particular Cases

(i) The vector field v under consideration is a parallel vector field of F_{n-1}. Putting $\frac{\delta v^\alpha}{\delta s} = 0$ in (3.7) we find that the equation is satisfied identically. Hence, we have the following theorem:

Theorem 3.4—If a vector field undergoes a parallel displacement along a curve C, then C is a secondary K_λ curve of the vector field relative to any congruence.

(ii) The vector field is tangential to the curve C. The equation of the secondary K_λ curve of F_{n-1} becomes

$$ p^\gamma - g_{\alpha\beta}t^{*\beta}p^\alpha(1-C^{*2}\psi)^{-1}t^{*\gamma} = 0. \quad .. \quad .. \quad .. \quad (3.12) $$

4. UNION CURVES AND K_λ CURVES

We shall find the condition under which the union curve (or secondary union curve) of a vector field (Nisha Rani 1968) be a K_λ curve (or secondary K_λ curve) of the vector field.

For a union curve of the vector field (relative to a congruence λ), we have

$$\lambda^i = Av^\alpha X^i_\alpha + B\frac{\delta v^i}{\delta s}. \quad .. \quad .. \quad .. \quad .. \quad (4.1)$$

The K_λ curve of the vector field relative to the same congruence λ^i is given by

$$\lambda^i = u\frac{\delta v^\alpha}{\delta s}X^i_\alpha + w\frac{\delta v^i}{\delta s}. \quad .. \quad .. \quad .. \quad .. \quad (4.2)$$

Comparing eqns. (4.1) and (4.2), we have

$$Av^\alpha = u\frac{\delta v^\alpha}{\delta s}, \quad w = B. \quad .. \quad .. \quad .. \quad (4.3)$$

Assuming that λ^i is not in the direction of $\frac{\delta v^i}{\delta s}$ (i.e. $A \neq 0$, $u \neq 0$) we have the following theorem:

Theorem 4.1—If a K_λ curve of a vector field is a union curve of the vector field then the derived vector (with respect to the subspace) in the direction of the curve is along the vector field itself.

The corresponding property holds for the secondary K_λ curve of the vector field also.

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