ON AXIALLY SYMMETRIC SELF-SUPERPOSABLE FLOWS OF THE TYPE CURL $\vec{H}_1 = \lambda_2 \vec{H}_2$ AND CURL $\vec{H}_2 = \lambda_1 \vec{H}_1$ IN MAGNETOHYDRODYNAMICS

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The purpose of the present paper is to find the most general flows of the type

$$\text{curl } \vec{H}_1 = \lambda_2 \vec{H}_2$$

and

$$\text{curl } \vec{H}_2 = \lambda_1 \vec{H}_1$$

in magnetohydrodynamics when the flows are self-superposable in inviscid and incompressible fluids and when λ_1 and λ_2 are functions of r and t alone. Further we find the most general flows of the given type when λ_1 and λ_2 are functions of z and t alone.

1. INTRODUCTION

Kapur (1960) discussed axially symmetric flows of the type

$$\text{curl } \vec{q}_1 = \lambda_2 \vec{q}_2$$

and

$$\text{curl } \vec{q}_2 = \lambda_1 \vec{q}_1$$

when the velocity vectors have both poloidal and toroidal components. Singh (1959) studied proportionately factors λ_1 and λ_2 in homogeneous viscous incompressible fluids of the type

$$\text{curl } \vec{q}_1 = \lambda_2 \vec{q}_2$$

and

$$\text{curl } \vec{q}_2 = \lambda_1 \vec{q}_1$$

under the assumption

$$\lambda_1 = \frac{\lambda_2}{\alpha^2}$$

where

$$\alpha = \int \lambda_2 \, dz.$$

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It has been proved that λ_1 and λ_2 can be functions of z and t only. He obtained expressions for velocities $\vec{q}_1$ and $\vec{q}_2$. The components of these velocity vectors along the direction of z-axis come out equal to zero. Strang (1948) has taken a more general case in which he assumes

$$\text{curl } \vec{q}_1 = \lambda_1 \vec{q}_1 + \mu_1 \vec{q}_2$$

$$\text{curl } \vec{q}_2 = \lambda_2 \vec{q}_2 + \mu_2 \vec{q}_1 .$$

Ghildyal (1957) proved that uniplaner superposable flows of the type

$$\text{curl } \vec{q}_1 = \lambda_2 \vec{q}_2$$

and

$$\text{curl } \vec{q}_2 = \lambda_1 \vec{q}_1$$

exist only if λ_1 and λ_2 are absolute constants. Bhatia (1964) established a theorem giving the condition for both the flows of type

$$\text{curl } \vec{q}_1 = \lambda_2 \vec{q}_2$$

and

$$\text{curl } \vec{q}_2 = \lambda_1 \vec{q}_1$$

to be self-superposable when λ_1 and λ_2 are functions of one space variable and time only. He proved that axially symmetric flows of the given type in the case of inviscid, incompressible fluids are, however, not possible when λ_1 and λ_2 are functions of r and t only. But they are possible when λ_1 and λ_2 are functions of r and z respectively and have been characterized there.

In the present paper we have extended this type of flow to the more general case of hydromagnetics. We have obtained the most general flows of the given type in incompressible and inviscid flows.

2. Axially Symmetric Flows of the Given Type when λ_1 and λ_2 are Functions of r, t Alone

Let

$$\vec{q}_i = -\frac{1}{r}\frac{\partial \psi_i}{\partial z}\vec{i}_r + \frac{\Omega_i}{r}\vec{i}_\theta + \frac{1}{r}\frac{\partial \psi_i}{\partial r}\vec{i}_z \quad .. \quad .. \quad .. \quad (1)$$

$$\vec{H}_i = -\frac{1}{r}\frac{\partial \bar{\psi}_i}{\partial z}\vec{i}_r + \frac{\bar{\Omega}_i}{r}\vec{i}_\theta + \frac{1}{r}\frac{\partial \bar{\psi}_i}{\partial r}\vec{i}_z \quad .. \quad .. \quad .. \quad (2)$$

$$(i = 1, 2)$$

where (r, θ, z) defines a system of cylindrical coordinates, $\vec{i}_r$, $\vec{i}_\theta$, $\vec{i}_z$ are unit vectors along the three principal directions and ψ, Ω, $\bar{\psi}$ and $\bar{\Omega}$ are four scalars.

From (1) and (2) we have

$$\text{curl } \vec{q_i} = -\frac{1}{r}\frac{\partial \Omega_i}{\partial z}\vec{i_r} - \frac{1}{r} D^2\psi_i \vec{i_\theta} + \frac{1}{r}\frac{\partial \Omega_i}{\partial r}\vec{i_z} \quad .. \quad .. \quad (3)$$

$$\text{curl } \vec{H_i} = -\frac{1}{r}\frac{\partial \bar{\Omega}_i}{\partial z}\vec{i_r} + \frac{1}{r} D_2\bar{\psi}_i \vec{i_\theta} + \frac{1}{r}\frac{\partial \bar{\Omega}_i}{\partial r}\vec{i_z} \quad .. \quad .. \quad (4)$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad .. \quad .. \quad .. \quad (5)$$

The equations of magnetohydrodynamics in an incompressible viscous electrically conducting fluids are

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{H_i}}{\partial t^2} - \Delta^2 \vec{H_i} + \frac{4\pi\sigma\mu}{c^2}\left[\frac{\partial \vec{H_i}}{\partial t} + \text{curl } (\vec{H_i} \times \vec{q_i})\right] = 0 \quad .. \quad .. \quad (6)$$

$$\frac{\partial \vec{q_i}}{\partial t} - \vec{q_i} \times \text{curl } \vec{q_i} + \frac{\mu}{4\pi\rho}\vec{H_i} \times \text{curl } \vec{H_i} + \text{grad } \left(\tfrac{1}{2}\vec{q_i^2} + \frac{p}{\rho} - u_i\right) - \nu\Delta^2 \vec{q_i} = 0 \quad (7)$$

and

$$\text{div } \vec{q_i} = 0 \quad .. \quad .. \quad .. \quad .. \quad (8)$$

$$\text{div } \vec{H_i} = 0. \quad .. \quad .. \quad .. \quad .. \quad (9)$$

We know that

$$\text{div } (\text{curl } \vec{H_i}) = 0$$

that is

$$\text{div } (\text{curl } \vec{H_1}) = 0 \quad .. \quad .. \quad .. \quad .. \quad (10)$$

and

$$\text{div } (\text{curl } \vec{H_2}) = 0. \quad .. \quad .. \quad .. \quad .. \quad (11)$$

From the relations

$$\left.\begin{aligned} \text{curl } \vec{H_1} &= \lambda_2 \vec{H_2} \\ \text{curl } \vec{H_2} &= \lambda_1 \vec{H_1} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (12)$$

and from (10) we have

$$\text{div } (\lambda_2 \vec{H_2}) = 0$$

or

$$\text{div } (\lambda_2 \vec{H_2}) = \lambda_2 \text{ div } \vec{H_2} + \vec{H_2} \,.\, \text{grad } \lambda_2 = 0.$$

Using (9) this becomes

$$H_2 \,.\, \text{grad } \lambda_2 = 0. \quad .. \quad .. \quad .. \quad .. \quad (13)$$

Similarly

$$H_1 \,.\, \text{grad } \lambda_1 = 0. \quad .. \quad .. \quad .. \quad .. \quad (14)$$

From (2), (13) and (14) we have

$$\left(-\frac{1}{r}\frac{\partial\bar{\psi}_i}{\partial z}\right).\frac{\partial\lambda_i}{\partial r}=0$$

since

$$\frac{\partial\lambda_i}{\partial r}\neq 0.$$

Therefore

$$\frac{1}{r}\frac{\partial\bar{\psi}_i}{\partial z}=0$$

or

$$\left.\begin{aligned}\frac{\partial\bar{\psi}_1}{\partial z}&=0\\ \frac{\partial\bar{\psi}_2}{\partial z}&=0.\end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (15)$$

Hence both the magnetic vectors do not have radial components. Using the relations (1), (2), (4), (5) and (15) we have

$$\frac{\partial\bar{\Omega}_1}{\partial z}=0,\quad \frac{\partial\bar{\Omega}_2}{\partial z}=0 \quad .. \quad .. \quad .. \quad .. \quad (16)$$

$$\left.\begin{aligned}D^2\bar{\psi}_1=\frac{\partial^2\bar{\psi}_1}{\partial r^2}-\frac{1}{r}\frac{\partial\bar{\psi}_1}{\partial r}&=-\lambda_2\bar{\Omega}_2\\ \frac{\partial\bar{\Omega}_1}{\partial r}&=\lambda_2\frac{\partial\bar{\psi}_2}{\partial r}\end{aligned}\right\} \quad .. \quad .. \quad .. \quad (17)$$

$$\left.\begin{aligned}D^2\bar{\psi}_2=\frac{\partial^2\bar{\psi}_2}{\partial r}-\frac{\partial\bar{\psi}_2}{\partial r}&=-\lambda_1\bar{\Omega}_1\\ \frac{\partial\bar{\Omega}_1}{\partial r}&=\lambda\frac{\partial\bar{\psi}}{\partial r}.\end{aligned}\right\} \quad .. \quad .. \quad .. \quad (18)$$

Kapur (1959) obtained the conditions of self-superposability for axially symmetric flows

$$\frac{\partial(\Omega_i,\psi_i)}{\partial(r,z)}-\frac{\mu_e}{\rho}\frac{\partial(\bar{\Omega}_i,\bar{\psi}_i)}{\partial(r,z)}=0 \quad .. \quad .. \quad .. \quad (19)$$

$$\Omega_i\frac{\partial\bar{\psi}_i}{\partial r}-\bar{\Omega}_i\frac{\partial\psi_i}{\partial r}=0 \quad .. \quad .. \quad .. \quad .. \quad (20)$$

$$\Omega_i\frac{\partial\bar{\psi}_i}{\partial z}-\bar{\Omega}_i\frac{\partial\psi_i}{\partial z}=0 \quad .. \quad .. \quad .. \quad .. \quad (21)$$

$$\frac{\partial\left(\psi_i,\frac{1}{r^2}D^2\psi_i\right)}{\partial(r,z)}+\frac{1}{r^3}\frac{\partial}{\partial z}\Omega_i^2-\frac{\mu_e}{\rho}\left[\frac{\partial\left(\bar{\psi}_i,\frac{1}{r^2}D^2\bar{\psi}_i\right)}{\partial(r,z)}\right]+\frac{1}{r^3}\frac{\partial\bar{\Omega}^2}{\partial z}=0 \quad .. \quad (22)$$

and

$$\frac{\partial(\psi_i,\bar{\psi}_i)}{\partial(z,r)}=0. \quad .. \quad .. \quad .. \quad .. \quad (23)$$

Using (15) and (21) we have

$$\bar{\Omega}_i \frac{\partial \psi_i}{\partial z} = 0.$$

Since the given magnetic fields are purely axi-symmetric so

$$\bar{\Omega}_i \neq 0.$$

Therefore

$$\frac{\partial \psi_i}{\partial z} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

The equation of motion is given by

$$\frac{\partial \vec{q}_i}{\partial t} + \text{grad}\left(\tfrac{1}{2}\vec{q}_i^2 + \frac{p_i}{\rho} + u_i\right) + \left(\vec{w}_i \times \vec{q}_i\right) = \frac{\mu_e}{\rho}\vec{J}_i \times \vec{H}_i + \nu\nabla^2\vec{q}_i. \qquad \ldots \quad (25)$$

Taking curl of both sides

$$\frac{\partial \vec{w}_i}{\partial t} + \text{curl}\left(\vec{w}_i \times \vec{q}_i\right) = \frac{\mu_e}{\rho}\,\text{curl}\left(\vec{J}_i \times \vec{H}_i\right) + \nu\nabla^2\,\text{curl}\,\vec{q}_i.$$

Since the flows are self-superposable it becomes

$$\frac{\partial \vec{w}_i}{\partial t} = \nu\nabla^2\,\text{curl}\,\vec{q}_i.$$

For inviscid liquids it becomes

$$\frac{\partial \vec{w}_i}{\partial t} = 0$$

$$\text{or} \quad \frac{\partial}{\partial t}\left(\text{curl}\,\vec{q}_i\right) = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

Using (3) it becomes

$$\frac{\partial}{\partial t}\left(-\frac{1}{r}\frac{\partial \Omega_i}{\partial z}\right) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

$$\frac{\partial}{\partial t}\left(-\frac{1}{r}D^2\psi_i\right) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

and

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial \Omega_i}{\partial r}\right) = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

From (28) we have

$$\frac{1}{r}D^2\psi_i = F_i(t)$$

$$\text{or} \quad \frac{1}{r}\left(\frac{\partial^2\psi_i}{\partial r^2} - \frac{1}{r}\frac{\partial \psi_i}{\partial r}\right) = F_i(t)$$

$$\text{or} \quad r\frac{\partial^2\psi_i}{\partial r^2}\frac{\partial \psi_i}{\partial r} = r^2F_i(t).$$

Integrating it

$$\left(\frac{r\partial\psi_i}{\partial r}\right) = r^3F_i(t)+K \quad .. \quad .. \quad .. \quad .. \quad (30)$$

where K is constant and $F_i(t)$ is arbitrary differentiable function of t.

From (29) we have

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial\Omega_i}{\partial r}\right) = 0$$

$$\text{or} \quad \Omega_i = f_i(t)+\phi_i(r) \quad .. \quad .. \quad .. \quad .. \quad (31)$$

where $f_i(t)$ and $\phi_i(r)$ are arbitrary differentiable functions of t and r respectively so that the general forms of velocity vectors are from (1), (24), (30) and (31), we have

$$\vec{q_1} = \frac{1}{r}[f_1(t)+\phi_1(r)]\vec{i_\theta}+\frac{1}{r}\left[r^2F_1(t)+\frac{k}{r}\right]\vec{i_z} \quad .. \quad .. \quad (32)$$

$$\vec{q_2} = \frac{1}{r}[f_2(t)+\phi_2(r)]\vec{i_\theta}+\frac{1}{r}\left[r^2F_2(t)+\frac{k}{r}\right]\vec{i_z}. \quad .. \quad .. \quad (33)$$

Bhatia (1964) obtained general forms of velocity vectors in ordinary fluids of the type

$$\text{curl}\ \vec{q_1} = \lambda_2\vec{q_2}$$

and

$$\text{curl}\ \vec{q_2} = \lambda_1\vec{q_1}$$

as

$$\vec{q_1} = \frac{1}{r}\{f_1(r)+\phi_1(t)\}\vec{i_\theta}+\frac{1}{r}\frac{f_2(r)}{\rho_1(r)}\{f_1(r)+\phi_1(t)\}\vec{i_z}$$

$$\vec{q_2} = \frac{1}{r}\{f_2(r)+\phi_2(t)\}\vec{i_\theta}+\frac{1}{r}\frac{f_1(r)}{\rho_2(r)}\{f_2(r)+\phi_2(t)\}\vec{i_z}.$$

3. Axially Symmetric Flows of the Given Type when λ_1 and λ_2 are Functions of z, t Alone

From (2) and (12) we have

$$\frac{\partial\lambda_i}{\partial z}\left(\frac{1}{r}\frac{\partial\psi_i}{\partial r}\right) = 0 \quad .. \quad .. \quad .. \quad .. \quad (34)$$

since

$$\frac{\partial\lambda_i}{\partial z} \neq 0.$$

Hence

$$\frac{1}{r}\frac{\partial\psi_i}{\partial r} = 0. \quad .. \quad .. \quad .. \quad .. \quad (35)$$

From the relations (1), (2), (4), (5) and (35) we have

$$\frac{\partial \bar{\Omega}_1}{\partial r} = 0, \quad \frac{\partial \bar{\Omega}_2}{\partial r} = 0 \quad .. \quad .. \quad .. \quad .. \quad (36)$$

$$\frac{\partial^2 \bar{\psi}_1}{\partial z^2} = -\lambda_2 \bar{\Omega}^2, \quad \lambda_2 \frac{\partial \bar{\psi}_2}{\partial^2} = \frac{\partial \bar{\Omega}_1}{\partial^2} \quad .. \quad .. \quad .. \quad (37)$$

$$\frac{\partial^2 \bar{\psi}_2}{\partial z^2} = -\lambda_1 \bar{\Omega}_1, \quad \lambda_1 \frac{\partial \bar{\psi}_1}{\partial z} = \frac{\partial \bar{\Omega}_2}{\partial z}. \quad .. \quad .. \quad .. \quad (38)$$

On using (20) and (35)

$$\bar{\Omega}_i \frac{\partial \psi_i}{\partial r} = 0$$

since

$$\bar{\Omega}_i \neq 0.$$

Hence

$$\frac{\partial \psi_i}{\partial r} = 0. \quad .. \quad .. \quad .. \quad .. \quad (39)$$

From (28) and (39) we get

$$\frac{\partial}{\partial t}\left(-\frac{1}{r}\frac{\partial^2 \psi_i}{\partial z^2}\right) = 0$$

$$\text{or} \quad \frac{\partial^2 \psi_i}{\partial z^2} = r\,\Phi_i(t)$$

$$\text{or} \quad \frac{\partial \psi_i}{\partial z} = r\,z\,\Phi_i(t) + c \quad .. \quad .. \quad .. \quad .. \quad (40)$$

where $\Phi_i(t)$ is arbitrary differentiable function of (t) and c is constant.

From (27) we have

$$\frac{\partial}{\partial t}\left(-\frac{1}{r}\frac{\partial \Omega_i}{\partial z}\right) = 0$$

$$\text{or} \quad \Omega_i = \rho_i(z) + \chi_i(t) \quad .. \quad .. \quad .. \quad .. \quad (41)$$

where $\rho_i(z)$ and $\chi_i(t)$ are arbitrary differentiable functions of z and t respectively.

Hence the general forms of velocity vectors are, from (39), (40) and (41),

$$\vec{q}_1 = -\frac{1}{r}[r\,z\,\Phi_1(t) + c]\,\vec{i}_r + \frac{1}{r}[\rho_1(z) + \chi_1(t)]\,\vec{i}_\theta \quad .. \quad .. \quad (42)$$

$$\vec{q}_2 = -\frac{1}{r}[r\,z\,\Phi_2(t) + c]\,\vec{i}_r + \frac{1}{r}[\rho_2(z) + \chi_2(t)]\,\vec{i}_\theta. \quad .. \quad .. \quad (43)$$

Bhatia (1964) obtained the general forms of velocity vectors when λ_1 and λ_2 are functions of z and t as

$$\vec{q}_1 = -\frac{1}{r}\left[\frac{F_1(z) + \Phi_1(t)}{p_1(z)}\right] F_2(z)\,\vec{i}_r + \left[\frac{F_1(z) + \Phi_1(t)}{r}\right]\vec{i}_\theta$$

$$\vec{q}_2 = -\frac{1}{r}\left[\frac{F_2(z) + \Phi_2(t)}{p_2(z)}\right] F_1(z)\,\vec{i}_r + \left[\frac{F_2(z) + \Phi_2 t}{r}\right]\vec{i}_\theta.$$

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