COVERING CONSTANTS OF SOME NON-CONVEX DOMAINS *

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The covering constants of three non-convex polygonal regions in the plane and of pairs of equal intersecting circles for different positions of the centres are determined. All maximal covering lattices of these regions are also obtained.

§ 1. The subject of packings has been studied ever since the time of Gauss. One of the major developments in this field was the work of Minkowski in Geometry of Numbers. His fundamental theorem is a theorem on lattice packings of convex bodies. He also gave methods for finding the best lattice packings of two- and three-dimensional symmetric convex bodies. The concept of best lattice packings and that of critical determinants of related bodies are closely allied. Although critical determinants of convex bodies have been studied for a long time, until 1937 there was not much work on critical determinants of non-convex bodies. Since 1937, inspired by Davenport's work on critical determinants of the region $|xyz| < |$ and $|x(y^2+z^2)| < |$, a great deal of work has been done on non-convex bodies.

Although there are similarities between the concepts of lattice coverings and lattice packings, the theory of coverings has not been as well developed. The covering constant of a circle has been known for a long time (*see* Kershner 1939). Fary (1950) gave a method of determining the covering constant of a plane symmetrical convex domain. The covering constant of three-dimensional spheres was determined by Bambah (1953) and that of four-dimensional spheres by Deloné and Ryskov (1963) (*see also* Barnes 1956, Few 1956, Dickson 1967). It follows from a theorem of Féjes Tóth 1950 (*see also* Bambah and Rogers 1952, Bambah *et al.* 1964, Bambah and Woods 1968) that if K is a symmetrical convex domain and B is an n-dimensional unit box, then the covering constant of $K \times B$ equals that of K. These seem to be the only bodies other than the obvious space-filling ones whose covering constants are known. A great deal of work has also been done on finding upper and lower estimates for the covering constants of n-dimensional convex bodies and of

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n-dimensional spheres. Not much work has been done on lattice coverings of non-convex bodies. Some results of Cassels (1952), Davenport (1950*a*, *b*, 1951), Ennola (1958), Pitman (1959) are equivalent to some upper and lower estimates for the covering constants of $|xy| \leqslant 1$, $|x(y^2+z^2)| \leqslant 1$ and $|(x^2+y^2)(z^2+u^2)| \leqslant 1$. These results have been proved by arithmetic methods.

The object of this paper is to find the best lattice coverings of some two-dimensional non-convex domains (critical determinants of a large number of special non-convex domains were determined in the forties and fifties under the inspiration of Mordell, Mahler and Davenport). We apply what can be called a localization method to determine the covering constants of three-polygonal regions and pairs of equal intersecting circles for different positions of the centres. We also determine all maximal covering lattices of these domains. This method can be compared to Mordell's method for finding the critical determinants of plane non-convex domains (*see* Cassels 1959).

It is hoped that these results may be useful for discussion of various problems on coverings by non-convex bodies, e.g. it may be possible to settle whether there exist some non-convex domains for which the best general covering is better than the best lattice covering.

§ 2. *Definitions and a Basic Lemma*

Let $A_1, \ldots, A_n$ be n linearly independent points in R_n, the n-dimensional Euclidean space. The set $\Lambda = \{u_1 A_1 + \ldots + u_n A_n : u_1, \ldots, u_n \text{ integers}\}$ is called a lattice and $A_1, \ldots, A_n$ is called a basis of Λ. If $A_i = (a_{i1}, \ldots, a_{in})$ then $d(\Lambda) = |\det(a_{ij})|$ is called the determinant of the lattice Λ; it is independent of the choice of a basis of Λ.

Let S be a set in R_n. A lattice Λ is said to be a 'covering lattice' for S if for every point X in R_n there is a point A in Λ such that X lies in the translate of S through A. The covering constant $c(S)$ of S is defined by $c(S) = \sup d(\Lambda)$ where Λ runs over all covering lattices for S (define $c(S) = 0$ if S has no covering lattice). A covering lattice Λ of S with $d(\Lambda) = c(S)$ is called a 'maximal covering lattice' for S.

The following lemma can be easily proved:

Lemma A—Let S be a set in R_n. Let V denote the volume of S. Let Λ be a covering lattice for S. Let $A \in \Lambda$, $A \neq O$, be such that the volume of $S \cap S+A$ is V'. Then

$$d(\Lambda) \leqslant V - V'.$$

If S is a plane set then $a(S)$ shall denote the area of S.

§ 3. *Theorem 1*—Let

$$S: \begin{cases} \max\{|x|, |y|\} < \frac{3}{2} \\ \min\{|x|, |y|\} < 1 \end{cases} \qquad \text{(see Fig. 1).}$$

Then $c(S) = 15/2$. The maximal covering lattices of S are of the type $T\Lambda$, where T is an automorphism of S and Λ is a lattice with basis $(0, 3)$ and $(5/2, \eta)$, $1 < \eta < 2$.

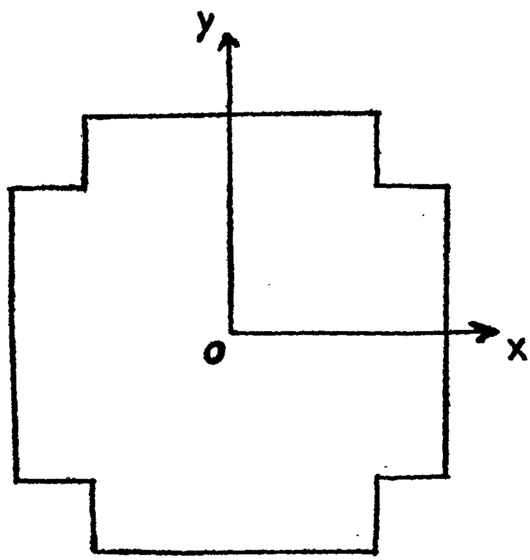

FIG. 1

PROOF: It is easy to see that the lattices generated by $(0, 3)$ and $(5/2, \eta)$, $1 \leqslant \eta \leqslant 2$, are covering lattices for S. Each of them has determinant $15/2$. So it is enough to prove that if Λ is any covering lattice for S then $d(\Lambda) \leqslant 15/2$ and equality occurs only if Λ is one of the lattices stated in the theorem.

Let Λ be a covering lattice for S. On considering the points near $P = (0, 3/2)$ we see that there is an $A \in \Lambda$, $A \neq O$ such that $P \in S+A$. Therefore $A \in -S+P = S+P$. Let $A = (a_1, a_2)$. Then either $-3/2 \leqslant a_1 \leqslant 3/2$ and $1/2 \leqslant a_2 \leqslant 5/2$ or $-1 < a_1 < 1$ and $0 \leqslant a_2 \leqslant 3$.

If $a_1 > 0$, the automorphism $x \to -x$ of S sends $A = (a_1, a_2)$ to a point $(-a_1, a_2) \in S+P$, and the lattice Λ to a lattice Λ' with $d(\Lambda') = d(\Lambda)$ which is also a covering lattice for S. Replacing Λ by Λ' and A by $(-a_1, a_2)$, if necessary, we can suppose that $a_1 \leqslant 0$. Hence A lies in the region

$$R = \left\{(a_1, a_2) : \begin{array}{l} \text{either } -\frac{3}{2} \leqslant a_1 < 0 \text{ and } \frac{1}{2} \leqslant a_2 < \frac{5}{2} \\ \text{or } \quad -1 < a_1 \leqslant 0 \text{ and } 0 < a_2 \leqslant 3 \end{array}\right\} \quad \text{(shaded region in Fig. 2).}$$

If $-3/2 < a_1 < -1$ and $1/2 < a_2 \leqslant 2+1/4$, then $a(S \cap S+A) > 1/2$, since the minimum overlap occurs only for $A = (-3/2, 9/4)$ where it is greater than $1/2$.

If $-1 < a_1 < 0$ and $0 < a_2 < 5/2$, then also $a(S \cap S+A) \geqslant 1/2$, since the minimum overlap occurs only for $A = (-1, 5/2)$ where it is $1/2$.

Now $a(S) = 8$, therefore, by Lemma A, we get $d(\Lambda) \leqslant 8-1/2 = 15/2$ and inequality will hold unless $A = (-1, 5/2)$.

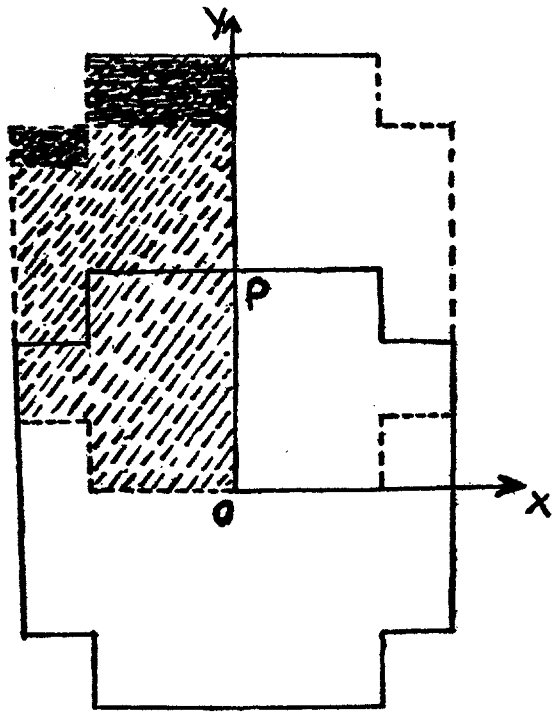

FIG. 2

It remains to consider the case when $A \in R_1 \cup R_2$ where

$$R_1 = \{(a_1, a_2)\colon\ -1 \leqslant a_1 \leqslant 0 \text{ and } 5/2 < a_2 \leqslant 3\}\cdot$$

$$R_2 = \{(a_1, a_2)\colon\ -3/2 \leqslant a_1 \leqslant -1 \text{ and } 9/4 \leqslant a_2 \leqslant 5/2\}$$

(heavily shaded regions in Fig. 2).

If A is a non-primitive point of Λ, then the primitive point of Λ on OA is in $R-(R_1 \cup R_2)$ and the previous discussion applies. So we can suppose that A is a primitive point of Λ in $R_1 \cup R_2$.

Case (*i*)—Let $A \in R_1$. Then the point $Q = (a_1+1, 3/2)$ lies on the boundary of $S \cup S+A$. Therefore, there is a point $B \in \Lambda$, $B \neq O$ or A, such that $Q \in S+B$, i.e. $B \in S+Q$. Let $B = (b_1, b_2)$. Then either $a_1-1/2 \leqslant b_1 \leqslant a_1+5/2$ and $1/2 \leqslant b_2 \leqslant 5/2$ or $a_1 \leqslant b_1 \leqslant a_1+2$ and $0 \leqslant b_2 \leqslant 3$.

Since O and A lie on the boundary of $S+Q$ in such a way that the part of the ray OA common with $S+Q$ is the segment OA and because A is primitive, therefore B is linearly independent of A.

If $a_1-1/2 \leqslant b_1 \leqslant a_1+1$ and $0 \leqslant b_2 \leqslant 3/2$ then $a(S \cap (S+B)) > 1/2$.

If $a_1-1/2 \leqslant b_1 \leqslant a_1+1$ and $3/2 \leqslant b_2 \leqslant 3$ then $a((S+A) \cap (S+B)) > 1/2$.

By Lemma A, $d(\Lambda) < 15/2$, in these cases.

If $a_1+1 \leqslant b_1 \leqslant a_1+2$ and $0 \leqslant b_2 \leqslant 3$, then

$$0 < b_1a_2-b_2a_1 \leqslant (a_1+2)\ a_2-3a_1$$
$$\leqslant (a_1+2)\ 3-3a_1 = 6.$$

If $a_1+2 \leqslant b_1 \leqslant a_1+5/2$ and $1/2 \leqslant b_2 \leqslant 5/2$, then

$$0 < b_1a_2-b_2a_1 \leqslant (a_1+5/2)\ 3+(-a_1)\ 5/2 = 15/2+1/2a_1 \leqslant 15/2$$

and equality holds if $a_1 = 0$, $a_2 = 3$, $b_1 = a_1+5/2 = 5/2$ and $1/2 \leqslant b_2 \leqslant 5/2$.

Therefore, $d(\Lambda) \leqslant 15/2$ in each case, and equality holds if Λ is generated by A and B and $a_1 = 0$, $a_2 = 3$, $b_1 = 5/2$ and $1/2 \leqslant b_2 \leqslant 5/2$.

It can be easily seen that if Λ is generated by $A = (0, 3)$ and $B = (5/2, \lambda)$ for $1/2 \leqslant \lambda < 1$ or $2 < \lambda \leqslant 5/2$ then Λ is not a covering lattice for S.

Case (ii)—Let $A \in R_2$.

Now the point $Q = (a_1+3/2, 3/2)$ lies on the boundary of $S \cup S+A$. As before we have a point $B \in \Lambda \cap (S+Q)$, $B \neq O$ or A. Let $B = (b_1, b_2)$. $S+Q = R_3 \cup R_4$,
where

$$R_3 = \{(b_1, b_2):\ a_1+1/2 \leqslant b_1 \leqslant a_1+5/2 \text{ and } 0 \leqslant b_2 \leqslant 3\}$$
$$R_4 = \{(b_1, b_2):\ a_1 \leqslant b_1 \leqslant a_1+3 \text{ and } 1/2 \leqslant b_2 \leqslant 5/2\}.$$

As in Case (i), B is linearly independent of A.

If $B \in R_3$, then

$$b_1a_2-b_2a_1 \leqslant (a_1+5/2)\ 5/2+3\ (-a_1)$$
$$= 25/4+(-a_1)\ 1/2$$
$$\leqslant 25/4+3/4 = 7$$

and

$$b_1a_2-b_2a_1 \geqslant (a_1+1/2)a_2 \geqslant -5/2.$$

If $B \in R_4$, then

$$b_1a_2-b_2a_1 \leqslant (a_1+3)\ 5/2+(-a_1)\ 5/2 = 15/2$$

and

$$b_1a_2-b_2a_1 \geqslant a_1a_2+(-a_1)\ 1/2 \geqslant -3.$$

Therefore, in each case $d(\Lambda) \leqslant |b_1a_2-b_2a_1| \leqslant 15/2$ and $d(\Lambda) = 15/2$, if Λ is generated by A and B where $a_2 = 5/2$, $b_1 = a_1+3$, $b_2 = 5/2$, i.e. Λ is generated by the points $(a_1, 5/2)$ and $(3, 0)$, $-3/2 \leqslant a_1 \leqslant -1$. These are of the type stated in the theorem.

§ 4. Here we give the covering constants of two other polygonal star regions. Since the techniques used to obtain them are the same as in § 3, we omit the proofs.

Theorem 2—Let S be the symmetric star domain shown in Fig. 3. Then $c(S) = 8$. The maximal covering lattices of S are of the type $T\Lambda_i$, $i = 1$ or 2, where T is an automorphism of S, Λ_1 is the lattice generated by the points (1, 3), (−2, 2) and Λ_2 is the lattice generated by the points (2, −2) and (2, 2).

Theorem 3—Let S be the symmetric star domain shown in Fig. 4. Then $c(S) = 13$ and the maximal covering lattices of S are given by $T\Gamma$ where T is an automorphism of S and Γ is the lattice generated by the points (3, −1) and (4, 3).

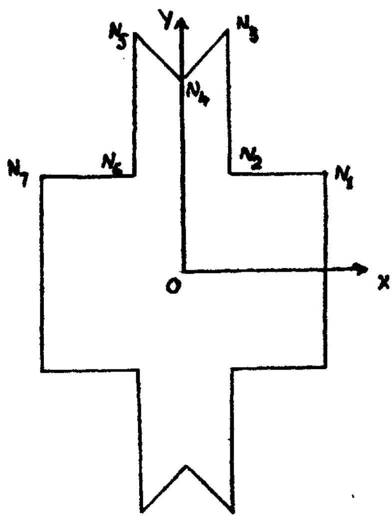

FIG. 3. $N_1 = (3/2, 1)$, $N_2 = (1/2, 1)$, $N_3 = (1/2, 5/2)$, $N_4 = (0, 2)$, $N_5 = (-1/2, 5/2)$, $N_6 = (-1/2, 1)$, $N_7 = (-3/2, 1)$.

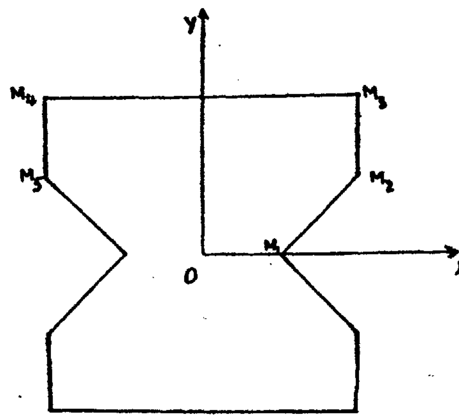

FIG. 4. $M_1 = (1, 0)$, $M_2 = (2, 1)$, $M_3 = (2, 2)$, $M_4 = (-2, 2)$, $M_5 = (-2, 1)$.

§ 5. *Theorem 4*—Let

$$S_k\colon \min\{(x-k)^2+y^2,\ (x+k)^2+y^2\} \leqslant 1.$$

(i) If $0 < k \leqslant \dfrac{\sqrt{17}-1}{8}$ then

$$c(S_k) = \gamma\left[2k+1+\tfrac{1}{2}\sqrt{4-\gamma^2}\right]$$

where

$$\gamma = \left(\frac{4-(2k+1)^2+(2k+1)\sqrt{(2k+1)^2+8}}{2}\right)^{\frac{1}{2}}.$$

For $k > \dfrac{\sqrt{17}-1}{8}$ we define

$$f_k(\theta) = 2k\sqrt{1-k^2}+2k\sin\theta+\sin(\theta+\alpha)+2\sin\tfrac{1}{2}(\theta+\alpha)$$

where

$$0 \leqslant \alpha < \pi/2 \text{ and } \tan\alpha = \frac{\sqrt{1-k^2}}{k}.$$

We also define β by

$$\tan\beta = \frac{\sqrt{3}k+\sqrt{1-k^2}}{-k+\sqrt{3}\sqrt{1-k^2}} \text{ and } 0 < \beta < \pi.$$

(ii) If $\frac{\sqrt{17}-1}{8} \leqslant k < 1/2$, then

$$c(S_k) = \max_{\alpha^* \leqslant \theta \leqslant \pi/2} f_k(\theta).$$

(iii) If $1/2 \leqslant k \leqslant \sqrt{3}/2$, then

$$c(S_k) = \max_{\beta \leqslant \theta \leqslant \pi/2} f_k(\theta).$$

(iv) If $\sqrt{3}/2 \leqslant k \leqslant 1$, then

$$c(S_k) = \max_{\pi/2 \leqslant \theta \leqslant \beta} f_k(\theta).$$

In each case maximum is attained for a unique $\theta = \alpha_k$ (say).

The maximal covering lattices of S_k are of the type $T\Lambda_k$, where T is an automorphism of S_k and Λ_k is generated by the points $A = (a_1, a_2)$ and

$$B = \left(x_1+k+\frac{a_2}{\sqrt{a_1^2+a_2^2}},\ y_1-\frac{a_1}{\sqrt{a_1^2+a_2^2}}\right)$$

where, in Case (i),

$$a_1 = 0,\ a_2 = \gamma$$

$$x_1 = k+\tfrac{1}{2}\sqrt{4-\gamma^2},\ y_1 = \tfrac{1}{2}\gamma.$$

In all other cases,

$$a_1 = -k+\cos\alpha_k, \quad a_2 = \sqrt{1-k^2}+\sin\alpha_k$$

$$x_1 = k+\cos\alpha_k, \qquad y_1 = \sin\alpha_k.$$

Proof: We first prove that the determinant of any covering lattice of S_k does not exceed the value of $c(S_k)$ stated in the theorem. We then prove the existence of covering lattices of S_k with this determinant and determine all such lattices in the process.

Let Λ be any covering lattice for S_k. On considering the points near $P = (0, \sqrt{1-k^2})$ we find that P lies in a domain S_k+A, $A \neq O$, $A \in \Lambda$. Consequently, $A \in S_k+P$. Let $A = (a_1, a_2)$. Then

$$\min\{(a_1+k)^2+(a_2-\sqrt{1-k^2})^2,\ (a_1-k)^2+(a_2-\sqrt{1-k^2})^2\} \leqslant 1.$$

If $a_1 > 0$, then the automorphism $x \to -x$ of S_k sends the point A to the point $(-a_1, a_2) \in S_k+P$ and the lattice Λ to a lattice Λ' with $d(\Lambda') = d(\Lambda)$. Moreover, Λ' is a covering lattice for S_k. So replacing $A = (a_1, a_2)$ by $(-a_1, a_2)$ and Λ by Λ', if necessary, we can suppose that $a_1 \leqslant 0$. If $a_2 < 0$, then the automorphism $y \to -y$ of S_k sends A to the point $(a_1, -a_2)$ and the lattice Λ to a lattice Λ'', with $d(\Lambda'') = d(\Lambda)$, which is a covering lattice for S_k. On replacing A by $(a_1, -a_2)$ and Λ by Λ'', if necessary, we can further suppose that $a_2 \geqslant 0$.

So it is enough to consider A lying in the region R given by

$$R = \left\{ (a_1, a_2) : \begin{array}{c} (a_1+k)^2+(a_2-\sqrt{1-k^2})^2 \leqslant 1 \\ a_1 \leqslant 0,\ a_2 \geqslant 0 \end{array} \right\} \qquad \text{(see Fig. 5).}$$

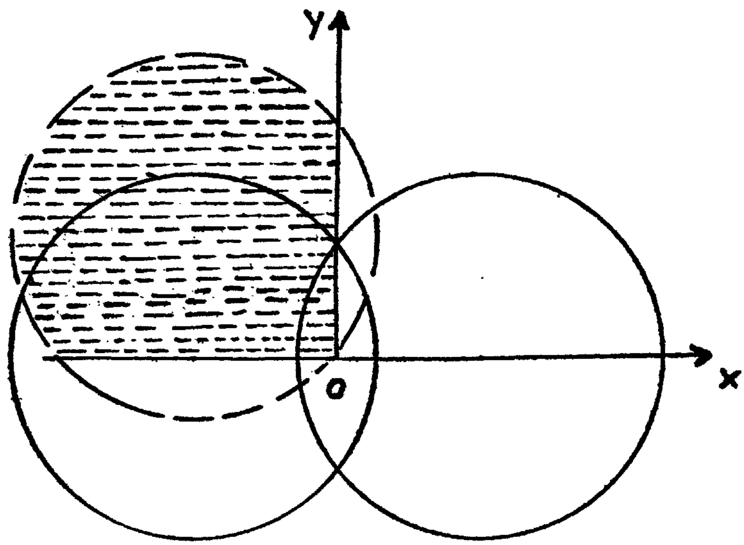

FIG. 5

R is convex and contains O on its boundary. . Therefore, if a point A_1 of Λ lies in R then so does the segment OA_1. By choosing the point of Λ nearest to the origin on the ray OA, we can suppose that A is a primitive point of Λ.

Now we notice that the point $Q = (x_1, y_1)$ where

$$x_1 = \tfrac{1}{2}(2k+a_1)+\tfrac{1}{2}a_2\sqrt{\frac{4-a_1^2-a_2^2}{a_1^2+a_2^2}}$$

$$y_1 = \tfrac{1}{2}a_2-\tfrac{1}{2}a_1\sqrt{\frac{4-a_1^2-a_2^2}{a_1^2+a_2^2}}$$

lies on the boundary of $S_k \cup (S_k+A)$. Therefore, there is a $B \in \Lambda$, $B \neq O$ or A such that $Q \in S_k+B$. Therefore, $B \in S_k+Q$. Let $B = (b_1, b_2)$.

Since Q lies on the 'right circles' of S_k and S_k+A, O and A lie on the 'left circle' of S_k+Q and B lies in S_k+Q. So the only part of the ray OA in S_k+Q is the segment OA. Since A is primitive, B is linearly independent of A.

Thus $d(\Lambda) \leqslant |b_1a_2-b_2a_1|$ and now we want to determine bounds for $b_1a_2-b_2a_1$.

We draw tangents to the circles

$$(x+k-x_1)^2+(y-y_1)^2 = 1$$

and

$$(x-k-x_1)^2+(y-y_1)^2 = 1$$

parallel to $a_2x-a_1y = 0$. These tangents are $a_2x-a_1y = \alpha_i$ where

$$\alpha_1 = (x_1-k)a_2-y_1a_1-\sqrt{a_1^2+a_2^2} = \tfrac{1}{2}\sqrt{a_1^2+a_2^2}\left(\sqrt{4-a_1^2-a_2^2}-2\right)$$

$$\alpha_2 = (x_1-k)a_2-y_1a_1+\sqrt{a_1^2+a_2^2} = \tfrac{1}{2}\sqrt{a_1^2+a_2^2}\left(\sqrt{4-a_1^2-a_2^2}+2\right)$$

$$\alpha_3 = (x_1+k)a_2-y_1a_1-\sqrt{a_1^2+a_2^2} = 2ka_2+\alpha_1$$

$$\alpha_4 = (x_1+k)a_2-y_1a_1+\sqrt{a_1^2+a_2^2} = 2ka_2+\alpha_2.$$

Since S_k+Q lies in the strip bounded by the two extreme tangents, therefore, we have

$$\alpha_1 \leqslant b_1a_2-b_2a_1 \leqslant \alpha_4.$$

Now $\alpha_4 \geqslant |\alpha_1|$. Therefore,

$$|b_1a_2-b_2a_1| \leqslant \alpha_4$$

where

$$\alpha_4 = 2ka_2+\tfrac{1}{2}\sqrt{a_1^2+a_2^2}\left(\sqrt{4-a_1^2-a_2^2}+2\right)$$

$$= \psi(a_1, a_2) \text{ (say)}$$

$$\alpha_4 \leqslant \max_{A\in R} \psi(a_1, a_2) = K \text{ (say)}.$$

We shall next show that K equals the value of $c(S_k)$ stated in the theorem.

Consider

$$\phi(z) = \sqrt{z}+\tfrac{1}{2}\sqrt{z(4-z)}, \quad 0 < z < 4.$$

Then

$$\phi'(z) = \tfrac{1}{2}z^{-\frac{1}{2}}\left[1+\frac{2-z}{\sqrt{4-z}}\right].$$

Thus $\phi'(z) \gtreqless 0$ according as $z \lesseqgtr 3$.

Therefore, $\phi(z)$ attains its maximum for $z = 3$.

Clearly, the circle $x^2+y^2 = 3$ meets the boundary of R in two points. Let $Q = (x', y')$ have the larger y-coordinate.

Since

$$\psi(a_1, a_2) = 2ka_2+\phi(a_1^2+a_2^2)$$

the maximum of $\psi(a_1, a_2)$ for $(a_1, a_2) \in R$ and $a_2 \leqslant y'$ occurs at $Q = (x', y')$.

Therefore,

$$K = \max_{(a_1, a_2)\in R_1} \psi(a_1, a_2)$$

where R_1 is the set of those A for which $a_2 \geqslant y'$.

We distinguish two cases.

Case I—$k \leqslant 1/2$.

Then $\sqrt{3} \leqslant 2\sqrt{1-k^2}$ and so $Q = (0, \sqrt{3})$.

Case II—$k \geqslant 1/2$.

Then $2\sqrt{1-k^2} < \sqrt{3}$ and Q lies on the boundary of the circle

$$(x+k)^2+(y-\sqrt{1-k^2})^2 = 1$$

and $x' < 0$.

Now we take up Case I.

Here

$$R_1 = \left\{ (a_1, a_2) : \begin{array}{c} (a_1+k)^2+(a_2-\sqrt{1-k^2})^2 < 1 \\ a_1 \leqslant 0,\ a_2 \geqslant \sqrt{3} \end{array} \right\}$$

$$\frac{\partial \psi}{\partial a_1} = \frac{a_1}{\sqrt{a_1^2+a_2^2}}\left[1+\frac{2-a_1^2-a_2^2}{\sqrt{4-a_1^2-a_2^2}}\right].$$

For $(a_1, a_2) \in R_1$, $\dfrac{\partial \psi}{\partial a_1} \geqslant 0$ and equality holds if $a_1 = 0$ and $a_2 = \sqrt{3}$.

Therefore for fixed a_2 the maximum of $\psi(a_1, a_2)$ for $A \in R_1$ occurs when a_1 is maximum. Thus

$$K = \max_{(a_1, a_2) \in \Gamma_1 \cup \Gamma_2} \psi(a_1, a_2)$$

where

$$\Gamma_1 = \{(a_1, a_2) : a_1 = 0,\ \sqrt{3} < a_2 < 2\sqrt{1-k^2}\}$$

$$\Gamma_2 = \left\{ (a_1, a_2) : \begin{array}{c} a_1 = -k+\cos\theta,\ a_2 = \sqrt{1-k^2}+\sin\theta \\ 0 < \alpha < \theta < \pi/2,\ \tan\alpha = \dfrac{\sqrt{1-k^2}}{k} \end{array} \right\}.$$

When $A \in \Gamma_1$,

$$\psi(a_1, a_2) = (2k+1)a_2+\tfrac{1}{2}a_2\sqrt{4-a_2^2}$$
$$= \psi(a_2) \quad \text{(say)}$$
$$\psi'(a_2) = (2k+1)+\frac{2-a_2^2}{\sqrt{4-a_2^2}}.$$

For $a_2 > 0$, $\psi'(a_2) \gtreqless 0$ according as $a_2 \lesseqgtr \gamma$ where

$$\gamma = \left(\frac{4-(2k+1)^2+(2k+1)\sqrt{8+(2k+1)^2}}{2}\right)^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad (5.1)$$

Therefore the maximum of $\psi(a_2)$ for $a_2 > 0$ occurs for $a_2 = \gamma$. Obviously $\gamma \geqslant \sqrt{3}$.

Now we want to determine those k for which $\gamma < 2\sqrt{1-k^2}$. On simplification we see that this happens if

$$\mu(k) = k^2(4k+5) < 1.$$

$\mu(k)$ is an increasing function of k for $k > 0$ and $\mu(k) = 1$ for $k = \frac{\sqrt{17}-1}{8}$. Therefore, it follows that

$$\gamma < 2\sqrt{1-k^2} \quad \text{if} \quad 0 < k \leqslant \frac{\sqrt{17}-1}{8}$$

$$\gamma \geqslant 2\sqrt{1-k^2} \quad \text{if} \quad \frac{\sqrt{17}-1}{8} \leqslant k \leqslant \tfrac{1}{2}.$$

Consequently, for

$$0 < k < \frac{\sqrt{17}-1}{8},$$

$$\max_{(a_1,\, a_2) \in \Gamma_1} \psi(a_1, a_2) = \psi(0, \gamma)$$

and for

$$\frac{\sqrt{17}-1}{8} \leqslant k \leqslant \tfrac{1}{2},$$

$$\max_{(a_1,\, a_2) \in \Gamma_1} \psi(a_1, a_2) = \psi\left(0, 2\sqrt{1-k^2}\right).$$

Now let $A \in \Gamma_2$. Then we have

$$\begin{aligned}\psi(a_1,a_2) &= 2ka_2+\sqrt{a_1^2+a_2^2}+\tfrac{1}{2}\sqrt{a_1^2+a_2^2}\sqrt{4-a_1^2-a_2^2}\\ &= 2k\sqrt{1-k^2}+2k \sin\theta+ \sin(\theta+\alpha)+2\sin\tfrac{1}{2}(\theta+\alpha)\\ &= f_k(\theta) \quad \text{(say)}.\end{aligned}$$

Here

$$0 < \alpha < \theta < \pi/2 \text{ and } \tan\alpha = \frac{\sqrt{1-k^2}}{k}$$

$$f_k'(\theta) = 2k\cos\theta+\cos(\theta+\alpha)+\cos\tfrac{1}{2}(\theta+\alpha)$$

$f_k'(\theta)$ decreases as θ increases between α and $\pi/2$. Therefore,

$$\begin{aligned}f_k'(\theta) < f_k'(\alpha) &= 2k\cos\alpha+\cos 2\alpha+\cos\alpha\\ &= 4k^2+k-1\\ &= 4\left(k-\frac{\sqrt{17}-1}{8}\right)\left(k+\frac{\sqrt{17}+1}{8}\right).\end{aligned}$$

Thus $f_k'(\alpha) \lesseqgtr 0$ for $k \lesseqgtr \frac{\sqrt{17}-1}{8}$. Therefore for $0 < k \leqslant \frac{\sqrt{17}-1}{8}$, $f_k(\theta)$ is a decreasing function of θ for $\alpha \leqslant \theta \leqslant \pi/2$. Therefore,

$$\begin{aligned}\max_{(a_1,\, a_2) \in \Gamma_2} \psi(a_1, a_2) &= \max_{\alpha < \theta < \pi/2} f_k(\theta)\\ &= f_k(\alpha) = \psi\left(0, 2\sqrt{1-k^2}\right) < \psi(0, \gamma).\end{aligned}$$

Hence in this case $\left(\text{i.e. when } 0 < k \leqslant \frac{\sqrt{17}-1}{8}\right)$, $K = \psi(0, \gamma)$ where γ is as given in (5.1).

When $\frac{\sqrt{17}-1}{8} \leqslant k \leqslant \frac{1}{2}$ then $f_k(\theta)$ attains its maximum for a unique $\theta = \alpha_k$, $\alpha \leqslant \alpha_k < \pi/2$. Therefore,

$$K = f_k(\alpha_k).$$

Case II—$1/2 \leqslant k \leqslant 1$.

The circles

$$x^2+y^2 = 3$$

and

$$(x+k)^2+(y-\sqrt{1-k^2})^2 = 1$$

intersect at the points $\left(\frac{-3k\pm\sqrt{3}\sqrt{1-k^2}}{2}, \frac{3\sqrt{1-k^2}\pm\sqrt{3}k}{2}\right)$ which are also the points of intersection of $x^2+y^2 = 3$ and boundary of R.

Therefore,

$$Q = (x', y') = \left(\frac{-3k+\sqrt{3}\sqrt{1-k^2}}{2}, \frac{3\sqrt{1-k^2}+\sqrt{3}k}{2}\right).$$

As remarked earlier

$$K = \max_{(a_1, a_2)\in R_1} \psi(a_1, a_2)$$

where

$$R_1 = R \cap \{(a_1, a_2)\colon a_2 \geqslant y'\}.$$

A simple calculation shows that

$$\text{if } \tfrac{1}{2} \leqslant k \leqslant \frac{\sqrt{3}}{2} \quad \text{then} \quad x' \geqslant -k$$

$$\text{and if } \frac{\sqrt{3}}{2} \leqslant k \leqslant 1 \quad \text{then} \quad x' \leqslant -k.$$

So we distinguish the following subcases:

Subcase (i): $\frac{1}{2} \leqslant k \leqslant \frac{\sqrt{3}}{2}$.

Since

$$x' \geqslant -k,\ a_1^2+a_2^2 \geqslant 3 \text{ for } (a_1, a_2) \in R_1$$

$$\frac{\partial\psi}{\partial a_1} = \frac{a_1}{\sqrt{a_1^2+a_2^2}}\left[1+\frac{2-a_1^2-a_2^2}{\sqrt{4-a_1^2-a_2^2}}\right].$$

Therefore, $\frac{\partial\psi}{\partial a_1} \geqslant 0$ for $A \in R_1$. **So for fixed a_2 the maximum of $\psi(a_1, a_2)$ occurs when a_1 is maximum.**

Therefore

$$K = \max_{(a_1, a_2) \in R_1} \psi(a_1, a_2) = \max_{(a_1, a_2) \in \Gamma} \psi(a_1, a_2)$$

where

$$\Gamma = \left\{ (a_1, a_2) : \begin{array}{c} (a_1+k)^2+(a_2-\sqrt{1-k^2})^2 = 1 \\ a_1 > -k,\ a_2 \geqslant y' \end{array} \right\}.$$

On Γ,

$$\begin{aligned} \psi(a_1, a_2) &= 2k\sqrt{1-k^2}+2k \sin \theta+\sin (\theta+\alpha)+2 \sin \tfrac{1}{2}(\theta+\alpha) \\ &= f_k(\theta), \quad 0 < \beta \leqslant \theta \leqslant \pi/2 \end{aligned}$$

where

$$\tan \beta = \frac{\sqrt{3}k+\sqrt{1-k^2}}{-k+\sqrt{3}\sqrt{1-k^2}}$$

$$f_k'(\theta) = 2k \cos \theta+\cos (\theta+\alpha)+\cos \tfrac{1}{2}(\theta+\alpha).$$

Since

$$\tan \alpha = \frac{\sqrt{1-k^2}}{k}, \text{ therefore, } 0 < \alpha < \pi/3.$$

So $f_k'(\theta)$ decreases as θ increases from β to $\pi/2$. Therefore, maximum of $f_k(\theta)$ for $\beta \leqslant \theta \leqslant \pi/2$ is attained for a unique $\theta = \alpha_k$ (say) in this interval. Therefore,

$$K = f_k(\alpha_k).$$

Subcase (ii): $\dfrac{\sqrt{3}}{2} < k < 1$.

Since $x' \leqslant -k$, $a_1^2+a_2^2 \leqslant 3$ for $A \in R_1$. Therefore $\dfrac{\partial \psi}{\partial a_1} \leqslant 0$. So for fixed a_2, the maximum of $\psi(a_1, a_2)$ for $A \in R_1$ occurs when a_1 is minimum. Therefore,

$$K = \max_{A \in R_1} \psi(a_1, a_2) = \max_{\pi/2 < \theta < \beta < \pi} f_k(\theta)$$

where again $\quad \tan \beta = \dfrac{\sqrt{3}k+\sqrt{1-k^2}}{-k+\sqrt{3}\sqrt{1-k^2}}$ and

$$f_k(\theta) = 2k\sqrt{1-k^2}+2k \sin \theta+\sin (\theta+\alpha)+2 \sin \tfrac{1}{2}(\theta+\alpha)$$

$$\cos (\alpha+\beta) = -\tfrac{1}{2}.$$

Therefore, $\alpha+\beta = 2\pi/3$. Thus $\theta+\alpha \leqslant 2\pi/3$ for $\theta \leqslant \beta$. Hence $f_k'(\theta)$ decreases as θ increases in $(\pi/2, \beta)$ and so maximum of $f_k(\theta)$ is attained for a unique $\theta = \alpha_k$ in this interval. Hence

$$K = f_k(\alpha_k).$$

Thus we have so far proved that if Λ is a covering lattice for S_k then $d(\Lambda)$ is at most equal to the value of $c(S_k)$ stated in the theorem. It only remains to prove that there do exist lattices with determinant equal to the stated value of $c(S_k)$ which gives a covering for S_k.

Let Λ_k be the lattice defined in the theorem.

Then our proof shows that $d(\Lambda_k) = c(S_k)$, for each k.

To prove that Λ_k is a covering lattice for S_k, it is enough to prove that the triangle OAB is covered by the domains S_k, S_k+A and S_k+B. For this it is enough to prove that

(*a*) the line OB lies between the points P_1 and P_2, these being the points of intersection of the circles

$$(x-k)^2+y^2 = 1$$

and

$$(x-b_1+k)^2+(y-b_2)^2 = 1$$

(*b*) the line AB lies between the two points of intersection of the circles

$$(x-k-a_1)^2+(y-a_2)^2 = 1$$

and

$$(x+k+b_1)^2+(y-b_2)^2 = 1$$

(*c*) the line OA lies between the two points of intersection of the circles

$$(x+k)^2+y^2 = 1$$

and

$$(x-k-a_1)^2+(y-a_2)^2 = 1.$$

Proof of (*a*): The line joining the centres $(k, 0)$ and (b_1-k, b_2) of the two circles lies between the points P_1 and P_2. Its mid-point $(\frac{1}{2}b_1, \frac{1}{2}b_2)$ is the same as the mid-point of the segment OB and so OB also lies between P_1 and P_2.

The proofs of (*b*) and (*c*) are similar.

It is clear from the proof that the maximal covering lattices are unique up to automorphisms.

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