ON THREE-PART MIXED BOUNDARY VALUE PROBLEMS IN TWO-DIMENSIONAL THEORY OF ELASTICITY

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The elastic equilibrium of a semi-infinite two-dimensional elastic medium containing two Griffith cracks situated parallel to the free boundary is investigated. Throughout, it is assumed that the equations of the classical theory of elasticity hold. In the case of an axisymmetric loading the problems are first reduced to a system of simultaneous triple integral equations involving trigonometric kernels. By the application of finite Hilbert transform these integral equations are reduced to simultaneous Fredholm integral equations which are finally solved iteratively. Analytical expressions for stress intensity factors, shape of the deformed crack and the energy required to open the cracks are derived.

1. Introduction

Recently, Srivastava and Lowengrub (1968) have developed a finite Hilbert transform technique for solving triple integral equations involving trigonometric kernels. Srivastava and Lowengrub (1968*a*, 1968, 1968*b*) have used this technique for the study of certain three-part mixed boundary value problems in infinite planes and infinitely long elastic strips. As a further demonstration of the effectiveness of finite Hilbert transform in the investigation of three-part mixed boundary value problems, we apply it for the study of the problem of finding the distribution of stress in a semi-infinite elastic medium containing two Griffith cracks situated in a line parallel to the free boundary (Fig. 1).

In the formulation of the problems we shall follow the method suggested by Kuz'min and Ufland (1965). Two problems in which different boundary conditions are satisfied are discussed. Each problem is first reduced to a system of simultaneous triple integral equations involving trigonometric kernel. By the application of the finite Hilbert transform technique developed in Srivastava and Lowengrub (1968) these triple integral equations are reduced to simultaneous Fredholm integral equations which are solved iteratively. To illustrate the application of the results derived in the following sections, expressions for stress intensity factors and the energy required to open the cracks, in the case when shearing stress on the surface of the cracks is zero and the normal stress is constant, are derived. The problem of a single Griffith crack has been dealt with as a limiting case of the problems discussed here. The results for single crack are in agreement with those derived in an earlier paper.

2. The Formulation of Problems

We shall consider displacement field in a perfectly elastic medium. With regards to its mechanical properties the medium is assumed to be isotropic and homogeneous. In the problems that we shall consider it is assumed that there is symmetry about the y-axis. For a symmetrical deformation the displacement vector U may be taken to have the components $(u, v, 0)$ and the components of stress tensor will be represented by σ_x, σ_y and τ_{xy}.

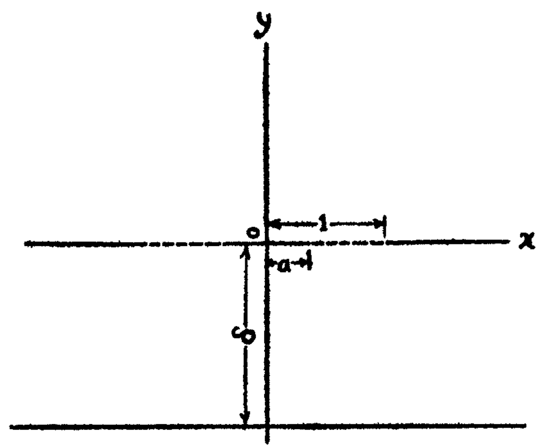

Fig. 1. Two Griffith cracks in a semi-infinite medium situated parallel to the free boundary

Let us assume that the medium is divided in two domains:

(1) The layer $-\delta < y < 0$ where δ is measured in terms of the far end of the cracks from origin which is taken to be the unit of measurement;

(2) The half space $0 < y < \infty$. Let the given stresses and components of displacement be denoted by indexes 1 and 2 in the two layers. We shall consider two problems:

Problem 1

The free boundary is assumed to be stress free and the stresses on the surface of cracks are prescribed. The boundary conditions can be written as

$$\sigma_y|_{y=-\delta} = 0, \qquad \tau_{xy}|_{y=-\delta} = 0, \text{ for all } x \qquad \text{(2.1)}$$

$$\left\{\begin{array}{ll} \sigma_y|_{y=0-} = \sigma_1(x), & \tau_{xy}|_{y=0-} = \tau_1(x) \\ \sigma_y|_{y=0+} = \sigma_2(x), & \tau_{xy}|_{y=0+} = \tau_2(x) \end{array}\right. \text{ for } a < |x| < 1. \qquad \text{(2.2)}$$

Problem 2

When the free boundary is rigidly clamped and the stresses are prescribed on the surface of the cracks. In this case in place of conditions (2.1) we have

$$u|_{y=-\delta} = 0, \quad v|_{y=-\delta} = 0, \text{ for all } x \qquad (2.3)$$

while conditions (2.2) are the same.

In addition, for $y = 0$, to pass through the region unoccupied by the cracks, the values of displacements and stress should be continuous. This requires the following additional conditions:

$$\left\{\begin{array}{ll} u|_{y=0+} = u|_{y=0-}, & \sigma_y|_{y=0+} = \sigma_y|_{y=0-} \\ v|_{y=0+} = v|_{y=0-}, & \tau_{xy}|_{y=0+} = \tau_{xy}|_{y=0-} \end{array}\right. \quad \begin{array}{l}\text{for } 0 < |x| < a \\ \text{and } 1 < |x| < \infty\end{array} \qquad (2.4)$$

It is assumed that $\delta \gg 1$.

3. The Equations of Equilibrium for the Elastic Field

The basic equations of equilibrium for two-dimensional elastic medium in the absence of body forces are:

$$2(1-\eta)\frac{\partial^2 u}{\partial x^2}+(1-2\eta)\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y} = 0 \qquad (3.1)$$

$$(1-2\eta)\frac{\partial^2 v}{\partial x^2}+2(1-\eta)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\,\partial y} = 0 \qquad (3.2)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad (3.3)$$

$$\sigma_y = \frac{2\mu}{1-2\eta}\left\{(1-\eta)\frac{\partial v}{\partial y}+\eta\frac{\partial u}{\partial x}\right\} \qquad (3.4)$$

where $\mu = E/2(1+\eta)$, η is Poisson's ratio and E the Young's modulus of the elastic material. Following Sneddon and Elliot (1946), for the solution of these partial differential equations we introduce Fourier sine and cosine transforms of u and v. We define

$$\bar{u}(\xi, y) = \mathfrak{F}_c[u(x, y), x \to \xi] \qquad (3.5)$$

$$\bar{v}(\xi, y) = \mathfrak{F}_s[v(x, y), x \to \xi]. \qquad (3.6)$$

Multiply (3.1) by $\sin \xi x$ and (3.2) by $\cos \xi x$ and integrate with respect to x from 0 to ∞ to get

$$[(1-2\eta)D^2-(2-2\eta)\xi^2]\bar{u}-\xi D\bar{v} = 0 \qquad (3.7)$$

$$[(2-2\eta)D^2-(1-2\eta)\xi^2]\bar{v}+\xi D\bar{u} = 0 \qquad (3.8)$$

where $D = d/dy$. From (3.3) and (3.4) we have

$$\bar{\tau}_{xy} = \mathfrak{F}_s[\tau_{xy}(x, y), x \to \xi] = \mu[D\bar{u}-\xi\bar{v}] \qquad (3.9)$$

$$\bar{\sigma}_y = \mathfrak{F}_c[\sigma_y(x, y), x \to \xi] = \frac{2\mu}{1-2\eta}[(1-\eta)D\bar{v}+\eta\xi\bar{u}]. \qquad (3.10)$$

4. Solutions for Semi-infinite Plane

In the case when the medium is free from disturbances at infinity, we are interested in the solutions which tend to zero as $y \to \infty$. Hence the solutions of (3.7) and (3.8) are

$$\bar{u} = (A+B\xi y)e^{-\xi y}, \quad \bar{v} = (A_1+B_1\xi y)e^{-\xi y} \qquad (4.1)$$

where

$$B = B_1,\ A_1 = A+(3-4\eta)B.$$

The expressions for the components of displacement vector and stress tensors are

$$u(x, y) = \mathcal{F}_s\Big[(A+B\xi y)e^{-\xi y},\ \xi \to x\Big] \qquad (4.2)$$

$$v(x, y) = \mathcal{F}_c\Big[(A_1+B\xi y)e^{-\xi y},\ \xi \to x\Big] \qquad (4.3)$$

$$\sigma_y(x, y) = -2\mu\mathcal{F}_c\Big[\{A+2(1-\eta)B+B\xi y\}\xi e^{-\xi y},\ \xi \to x\Big] \qquad (4.4)$$

$$\tau_{xy}(x, y) = -2\mu\mathcal{F}_s\Big[\{A+(1-2\eta)B+B\xi y\}\xi e^{-\xi y},\ \xi \to x\Big]. \qquad (4.5)$$

5. Solutions for the Layer $-\delta < y < 0$

The appropriate solutions in this case are

$$\bar{u} = \big[\{C+D\xi(y+\delta)\}\cosh \xi(y+\delta)+\{E+F\xi(y+\delta)\}\sinh \xi(y+\delta)/\sinh \xi\delta\big] \qquad (5.1)$$

$$\bar{v} = \big[\{C_1+D_1\xi(y+\delta)\}\cosh \xi(y+\delta)+\{E_1+F_1\xi(y+\delta)\}\sinh \xi(y+\delta)/\sinh \xi\delta\big] \qquad (5.2)$$

where

$$\left.\begin{aligned} D &= -F_1,\ (3-4\eta)\, D_1+C+ E_1 = 0 \\ D_1 &= -F,\ E+C_1-(3-4)\ D = 0 \end{aligned}\right\}. \qquad (5.3)$$

Problem 1

When free boundary is stress free we have to satisfy the conditions $\sigma_y = 0$, $\tau_{xy} = 0$ for $y = -\delta$. From (3.9) and (3.10) we have

$$\begin{aligned}\bar{\tau}_{xy} = \frac{\mu\xi}{\sinh \xi\delta}&[\{C+D(y+\delta)\xi+F-E_1-F_1\xi(y+\delta)\}\sinh \xi(y+\delta) \\ &+\{E+F(y+\delta)\xi+D-C_1-D_1\xi(y+\delta)\}\cosh \xi(y+\delta)] \end{aligned} \qquad (5.4)$$

$$\begin{aligned}\sigma_y = \frac{\mu\xi}{\sinh \xi\delta}&[\sinh \xi(y+\delta)\{(1-\eta)(C_1+D_1(y+\delta)\xi+F_1)+\eta(E+F(y+\delta)\xi)\} \\ &+\cosh \xi(y+\delta)\{(E_1+D_1+F_1(y+\delta)\xi)(1-\eta)+\eta(C+D(y+\delta)\xi)\}]. \end{aligned} \qquad (5.5)$$

The boundary conditions are satisfied provided

$$E+D-C_1 = 0 \qquad (5.6)$$

$$(1-\eta)(E_1+D_1)+\eta C = 0. \qquad (5.7)$$

These equations with (5.3) lead to the relations

$$\left.\begin{aligned} E &= (1-2\eta)D, & C_1 &= 2(1-\eta)D \\ E_1 &= -(1-2\eta)D_1, & C &= -2(1-\eta)D_1 \end{aligned}\right\}. \qquad (5.8)$$

Consequently, the components of displacement and stress tensor are

$$u(x, y) = \mathfrak{F}_s\Big[\Big(\cosh \xi(y+\delta)\{D(y+\delta)\xi-2(1-\eta)D_1\}+\sinh \xi(y+\delta) \times\{(1-2\eta)D-D_1(y+\delta)\xi\}\Big)/\sinh \xi\delta;\, \xi\to x\Big] \quad \ldots \quad (5.9)$$

$$v(x, y) = \mathfrak{F}_c\Big[\Big(\cosh \xi(y+\delta)\{2(1-\eta)D+D_1(y+\delta)\xi\}-\sinh \xi(y+\delta) \times\{D(y+\delta)\xi+(1-2\eta)D_1\}\Big)/\sinh \xi\delta;\, \xi\to x\Big] \quad \ldots \quad (5.10)$$

$$\sigma_y(x, y) = 2\mu\mathfrak{F}_c\Big[\Big(\sinh \xi(y+\delta)\{D+D_1(y+\delta)\xi\}-D(y+\delta)\xi \cosh \xi(y+\delta)\Big) \times\xi/\sinh \xi\delta;\, \xi\to x\Big] \quad \ldots \quad (5.11)$$

$$\tau_{xy}(x, y) = -2\mu\mathfrak{F}_s\Big[\Big(\sinh \xi(y+\delta)\{D_1-D(y+\delta)\xi\}+D_1(y+\delta)\xi \cosh \xi(y+\delta)\Big) \times\xi/\sinh \xi\delta;\, \xi\to x\Big]. \quad \ldots \quad (5.12)$$

Problem 2

In this case we have to satisfy the conditions $u = 0$, $v = 0$ for $y = -\delta$. These conditions imply that

$$C = C_1 = 0, \quad E = (3-4\eta)\, D, \quad E_1 = -(3-4\eta)D_1. \quad \ldots \quad (5.13)$$

Hence we have

$$u(x, y) = \mathfrak{F}_s\Big[\Big(D(y+\delta)\xi \cosh \xi(y+\delta)+\{(3-4\eta)D-D_1(y+\delta)\xi\} \times\sinh \xi(y+\delta)\Big)/\sinh \xi\delta;\, \xi\to x\Big] \quad \ldots \quad (5.14)$$

$$v(x, y) = \mathfrak{F}_c\Big[\Big(D_1(y+\delta)\xi \cosh \xi(y+\delta)-\{(3-4\eta)D_1+D(y+\delta)\xi\} \times\sinh \xi(y+\delta)\Big)/\sinh \xi\delta;\, \xi\to x\Big] \quad \ldots \quad (5.15)$$

$$\sigma_y(x, y) = -2\mu\mathfrak{F}_c\Big[\Big(\sinh \xi(y+\delta)\{(1-2\eta)D-D_1(y+\delta)\xi\}+\cosh \xi(y+\delta) \times\{D(y+\delta)\xi+D_1(2-2\eta)\}\Big)\xi/\sinh \xi\delta;\, \xi\to x\Big] \quad \ldots \quad (5.16)$$

$$\tau_{xy}(x, y) = -2\mu\mathfrak{F}_s\Big[\Big(\sinh \xi(y+\delta)\{D(y+\delta)\xi-D_1(1-2\eta)\}+\cosh \xi(y+\delta) \times\{D_1(y+\delta)\xi-2(1-\eta)D\}\Big)\xi/\sinh \xi\delta;\, \xi\to x\Big]. \quad \ldots \quad (5.17)$$

6. Reduction of the Problems to a System of Simultaneous Triple Integral Equations

Problem 1

We still have to satisfy the boundary conditions for $y = 0$. For the sake of simplicity we assume that $\tau_1(x) = \tau_2(x) = \tau(x)$ and $\sigma_1(x) = \sigma_2(x) = -p(x)$. For $a < x < 1$, we have

$$\int_0^\infty [D_1\xi\delta+D-D\xi\delta \coth \xi\delta]\xi \cos \xi x \, d\xi$$

$$= -\int_0^\infty [A+2(1-\eta)B]\xi \cos \xi x \, d\xi = -\frac{\pi P(x)}{8(1-\eta)} \quad \ldots \quad \ldots \quad (6.1)$$

$$\int_0^\infty [D_1-D\xi\delta+D_1\xi\delta \coth \xi\delta]\xi \sin \xi x \, d\xi$$

$$= \int_0^\infty [A+(1-2\eta)B]\xi \sin \xi x \, d\xi = -\frac{\pi T(x)}{8(1-\eta)} \quad \ldots \quad \ldots \quad (6.2)$$

where

$$P(x) = (2/\pi)^{\frac{1}{2}}\, 2(1-\eta)\, p(x)/\mu, \; T(x) = (2/\pi)^{\frac{1}{2}}\, 2(1-\eta)\, \tau(x)/\mu.$$

For $0 < x < a$ and $1 < x < \infty$, we have

$$\int_0^\infty [D_1\xi\delta+D-D\xi\delta \coth \xi\delta+A+2(1-\eta)B]\xi \cos \xi x \, d\xi = 0 \quad \ldots \quad \ldots \quad (6.3)$$

$$\int_0^\infty [D_1'-D\xi\delta+D_1\xi\delta \coth \xi\delta-A-(1-2\eta)B]\xi \sin \xi x \, d\xi = 0 \quad \ldots \quad \ldots \quad (6.4)$$

$$\int_0^\infty [\{D\xi\delta-2(1-\eta)D_1\} \coth \xi\delta+(1-2\eta)D-D_1\xi\delta-A] \sin \xi x \, d\xi = 0 \quad \ldots \quad (6.5)$$

$$\int_0^\infty [\{2(1-\eta)D+D_1\xi\delta\} \coth \xi\delta-D\xi\delta-D_1(1-2\eta)-A-(3-4\eta)B] \cos \xi x \, d\xi = 0. \quad \ldots \quad (6.6)$$

From (6.1) to (6.4) we have

$$Y = D(1-z \coth z)+D_1Z = -[A+2(1-\eta)B] \quad \ldots \quad (6.7)$$

$$Z = -Dz+D_1(1+z \coth z) = A+(1-2\eta)B. \quad \ldots \quad (6.8)$$

Let us write

$$M = D(z \coth z+1-2\eta)-D_1(z+(2-2\eta) \coth z)-A \quad (6.9)$$

$$N = D(2(1-\eta) \coth z-z)+D_1(z \coth z-1+2\eta)-A-(3-4\eta)B \quad (6.10)$$

where $z = \xi\delta$. With the help of the above equations we have

$$4(1-\eta)Y = N+I(z)N+J(z)M \quad \ldots \quad \ldots \quad (6.11)$$

$$-4(1-\eta)Z = M+K(z)M+L(z)N \quad \ldots \quad \ldots \quad (6.12)$$

where

$$I(z) = -(1+2z+2z^2)e^{-2z}, \; K(z) = -(1-2z+2z^2)e^{-2z}, \; J(z) = L(z) = -2z^2e^{-2z}.$$

Equations (6.1), (6.2), (6.5) and (6.6) together with (6.9) to (6.12) lead to a system of simultaneous triple integral equations

$$\left.\begin{aligned}\int_0^\infty M(\xi)\sin\xi x\,d\xi &= 0 \qquad (6.13)\\ \int_0^\infty N(\xi)\cos\xi x\,d\xi &= 0 \qquad (6.14)\end{aligned}\right\} 0<x<a,\ 1<x<\infty$$

$$\left.\begin{aligned}\int_0^\infty [N(\xi)+I(z)N(\xi)+M(\xi)J(z)]\xi\cos\xi x\,d\xi &= -\frac{\pi}{2}P(x) \qquad (6.15)\\ \int_0^\infty [M(\xi)+K(z)M(\xi)+L(z)N(\xi)]\xi\sin\xi x\,d\xi &= \frac{\pi}{2}T(x) \qquad (6.16)\end{aligned}\right\} a<x<1$$

Problem 2

The conditions (2.2) and (2.4) and exactly similar procedure, as for problem 1, lead to the system of simultaneous triple integral eqns. (6.13) to (6.16) with the functions I, J, K, L having the following values:

$$I(z) = 2(b+z+z^2)e^{-2z}/(3-4\eta), \quad K(z) = 2(b+z+z^2)e^{-2z}/(3-4\eta),$$

$$J(z) = L(z) = 2(z^2+c)e^{-2z}/(3-4\eta),$$

where $b = \frac{1}{2}(5-12\eta+\eta^2)$, $c = 2(1-\eta)(1-2\eta)$.

7. Some Useful Results

We list here for ready reference some integrals that we shall use in the future sections. All these are well-known results and can be found in Gradshteyn and Ryzhik (1965).

$$\int_0^\infty \sin xy\sin ax\frac{dx}{x} = \frac{1}{2}\log\left|\frac{a+y}{a-y}\right| \qquad (7.1)$$

$$\int_0^\infty \cos xy(1-\cos ax)\frac{dx}{x} = \frac{1}{2}\log\left|1-\frac{a^2}{y^2}\right| \qquad (7.2)$$

$$\int_a^b \{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\,dt = F\left[\frac{\pi}{2},\ q\right]/b = F/b \qquad (7.3)$$

$$\int_a^b t^2\{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\,dt = bE\left[\frac{\pi}{2},\ q\right] = bE \qquad (7.4)$$

where $q = 1-a^2/b^2$ and F and E are elliptic integrals of the first and second kind. It is quite simple to derive the following integrals:

$$\int_a^b \frac{tdt}{\{(t^2-a^2)(b^2-t^2)\}^{\frac{1}{2}}(t^2-x^2)} = \begin{cases}\pi/2\{(a^2-x^2)(b^2-x^2)\}^{\frac{1}{2}}, & 0<x<a\\ 0, & a<x<b\\ -\pi/2\{(x^2-a^2)(x^2-b^2)\}^{\frac{1}{2}}, & b<x<\infty\end{cases} \qquad (7.5)$$

$$\int_a^b \frac{(t^2-b^2E/F)\log|(t+x)/(t-x)|}{\{(t^2-a^2)(b^2-t^2)\}^{\frac{1}{2}}}\,dt = (x-\pi b/2F)\,,\ a<x<b \qquad (7.6)$$

$$\int_a^1 \frac{t(2t^2-a^2-1)\log|1-t^2/x^2|}{\{(t^2-a^2)(1-t^2)\}^{\frac{1}{2}}}\,dt = \frac{\pi}{2}(1+a^2-2x^2)\,.\ a<x<1. \qquad (7.7)$$

In addition, we shall use the following theorem on finite Hilbert transform given in Tricomi (1951).

If $p(y) \in L^2(a, b)$, then the integral equation

$$\frac{1}{\pi}\int_a^b \frac{h(x)}{x-y}\,dx = p(y),\ y \in (a, b) \qquad (7.8)$$

has the solution

$$h(x) = -\frac{1}{\pi}\int_a^b \left\{\frac{(x-a)(b-y)}{(b-x)(a-y)}\right\}^{\frac{1}{2}} \frac{p(y)}{y-x}\,dy + C/\{(x-a)(b-x)\}^{\frac{1}{2}} \qquad (7.9)$$

where C is an arbitrary constant.

Using the above theorem we find that the solution of the integral equation

$$\frac{2}{\pi}\int_a^1 \frac{xh(x^2)}{x^2-y^2}\,dx = p(y),\ y \in (a, 1) \qquad (7.10)$$

is given by

$$h(x^2) = -\mathfrak{H}[p(y)] + C/\{(x^2-a^2)(1-x^2)\}^{\frac{1}{2}} \qquad (7.11)$$

where

$$\mathfrak{H}[p(y)] = \frac{2}{\pi}\int_a^1 \left\{\frac{(x^2-a^2)(1-y^2)}{(1-x^2)(y^2-a^2)}\right\}^{\frac{1}{2}} \frac{y\,dy}{y^2-x^2}[p(y)].$$

8. Solution of the Simultaneous Triple Integral Equations Involving Trigonometric Kernels

We have to solve the integral equations

$$\left.\begin{aligned} &\int_0^\infty M(\xi)\sin\xi x\,d\xi = 0 \qquad (8.1)\\ &\int_0^\infty N(\xi)\cos\xi x\,d\xi = 0 \qquad (8.2)\end{aligned}\right\}\ 0<x<1,\ 1<x<\infty$$

$$\left.\begin{aligned} &\int_0^\infty [N(\xi)+I(\xi\delta)N(\xi)+J(\xi\delta)M(\xi)]\xi\cos\xi x\,d\xi = -\frac{\pi}{2}P(x) \qquad (8.3)\\ &\int_0^\infty [M(\xi)+K(\xi\delta)M(\xi)+L(\xi\delta)N(\xi)]\xi\sin\xi x\,d\xi = \frac{\pi}{2}T(x) \qquad (8.4)\end{aligned}\right\}\ a<x<1$$

where I, J, K, L, $P(x)$, $T(x)$ are known functions and $M(\xi)$ and $N(\xi)$ are unknown functions to be determined. For solving these equations we shall use finite Hilbert transform technique given by Srivastava and Lowengrub (1968).

The method of solution given here does not give a closed form of solution. It is, however, possible to reduce the above set of integral equations to a pair of simultaneous Fredholm integral equations of the second kind which can be solved by iterative method provided $\delta \gg 1$ and the integrals

$$\int_0^\infty u^n I(u)\, du,\; n = 0, 1, 2, \ldots$$

and similar integrals for J, K, L are convergent.

Let the trail solutions be

$$M(\xi) = \frac{1}{\xi}\int_a^1 \frac{g(t^2)}{t}(1-\cos \xi t)\, dt \qquad (8.5)$$

$$N(\xi) = \frac{1}{\xi}\int_a^1 h(t^2) \sin \xi t\, dt. \qquad (8.6)$$

Equations (8.1) and (8.2) are satisfied provided

$$\int_a^1 \frac{g(t^2)}{t}\, dt = 0, \quad \int_a^1 h(t^2)\, dt = 0. \qquad (8.7)$$

Equations (8.3) and (8.4) can be written as

$$\frac{2}{\pi}\frac{d}{dx}\int_0^\infty N(\xi) \sin \xi x\, d\xi = -P(x) - \frac{2}{\pi}\int_0^\infty [I(\xi\delta)N(\xi)+J(\xi\delta)M(\xi)]\xi \cos \xi x\, d\xi$$

$$\frac{2}{\pi}\frac{d}{dx}\int_0^\infty M(\xi) \cos \xi x\, d\xi = -T(x) + \frac{2}{\pi}\int_0^\infty [K(\xi\delta)M(\xi)+L(\xi\delta)N(\xi)]\xi \sin \xi x\, d\xi.$$

Substituting the value of $M(\xi)$ and $N(\xi)$, changing the order of integration and after using (7.1) and (7.2), we get

$$\frac{2}{\pi}\int_a^1 \frac{th(t^2)}{t^2-x^2}\, dt = -P(x) - \frac{2}{\pi}\int_a^1 h(t^2)\int_0^\infty I(\xi\delta) \sin \xi t \cos \xi x\, d\xi\, dt$$
$$-\frac{2}{\pi}\int_a^1 \frac{g(t^2)}{t}\int_0^\infty J(\xi\delta) \cos \xi x(1-\cos \xi t)\, d\xi\, dt \qquad (8.8)$$

$$\frac{2}{\pi}\int_a^1 \frac{tg(t^2)}{t^2-x^2}\, dt = xT(x) - \frac{2x}{\pi}\int_a^1 h(t^2)\int_0^\infty L(\xi\delta) \sin \xi x \sin \xi t\, d\xi\, dt$$
$$-\frac{2x}{\pi}\int_a^1 \frac{g(t^2)}{t}\int_0^\infty K(\xi\delta) \sin \xi x(1-\cos \xi t)\, d\xi\, dt. \qquad (8.9)$$

Using (7.11) we get

$$h(x^2) = \phi(x) + \int_a^1 h(t^2)R(t, x)\, dt + \int_a^1 \frac{g(t^2)}{t} S(t, x)\, dt + C/X \qquad (8.10)$$

$$g(x^2) = \psi(x) + \int_a^1 \frac{g(t^2)}{t} R_1(t, x)\, dt + \int_a^1 h(t^2) S_1(t, x)\, dt + C_1/X \qquad (8.11)$$

where C and C_1 are arbitrary constants, and

$$R(t, x) = \frac{2}{\pi} \mathfrak{H} \left[\int_0^\infty I(\xi\delta) \sin \xi t \cos \xi y \, d\xi \right], \quad X = \{(x^2-a^2)(1-x^2)\}^{\frac{1}{2}}$$

$$S(t, x) = \frac{2}{\pi} \mathfrak{H} \left[\int_0^\infty J(\xi\delta) \cos \xi y(1-\cos \xi t) \, d\xi \right], \quad \phi(x) = \mathfrak{H}[P(y)]$$

$$R(t, x) = \frac{2}{\pi} \mathfrak{H} \left[\int_0^\infty K(\xi\delta) \sin \xi y(1-\cos \xi t) \, d\xi \right], \quad \psi(x) = \mathfrak{H}[T(y)]$$

$$S_1(t, x) = \frac{2}{\pi} \mathfrak{H} \left[\int_0^\infty L(\xi\delta) \sin \xi y \sin \xi t \, d\xi \right].$$

Integrate (8.10) as such and (8.11) after dividing by x with respect to x from a to 1, because of (8.7) we get

$$0 = \int_a^1 \phi(y)\, dy + \int_a^1 h(t^2) \int_a^1 R(t, y)\, dy\, dt + \int_a^1 \frac{g(t^2)}{t} \int_a^1 S(t, y)\, dy\, dt + CF \tag{8.12}$$

$$0 = \int_a^1 \frac{\psi(y)}{y} dy + \int_a^1 \frac{g(t^2)}{t} \int_a^1 \frac{R_1(t, y)}{y} dy\, dt + \int_a^1 h(t^2) \int_a^1 \frac{S_1(t, y)}{y} dy\, dt + \frac{\pi C_1}{2a}. \tag{8.13}$$

Eliminating C, C_1 from (8.10), (8.11) and (8.12), (8.13) we get

$$h(x^2) = \phi_1(x) + \int_a^1 h(t^2)\chi(t, x)\, dt + \int_a^1 \frac{g(t^2)}{t} \Gamma(t, x)\, dt \quad .. \quad .. \tag{8.14}$$

$$g(x^2) = \psi_1(x) + \int_a^1 \frac{g(t^2)}{t} \chi_1(t, x)\, dt + \int_a^1 h(t^2)\Gamma_1(t, x)\, dt \quad .. \tag{8.15}$$

where

$$\phi_1(x) = \phi(x) - \frac{1}{FX} \int_a^1 \phi(y)\, dy, \quad \psi_1(x) = \psi(x) - \frac{2a}{\pi X} \int_a^1 \frac{\psi(y)\, dy}{y}$$

$$\chi(t, x) = R(t, x) - \frac{1}{FX} \int_a^1 R(t, y)\, dy$$

$$\Gamma(t, x) = S(t, x) - \frac{1}{FX} \int_a^1 S(t, y)\, dy$$

$$\chi_1(t, x) = R_1(t, x) - \frac{2a}{\pi X} \int_a^1 \frac{R_1(t, y)}{y} dy$$

$$\Gamma_1(t, x) = S_1(t, x) - \frac{2a}{\pi X} \int_a^1 \frac{S_1(t, y)}{y} dy.$$

In the above expressions one can make the substitution $\xi\delta = u$ and express sine and cosine functions in power series, after some calculations carried out with the help of results given in section 7, one gets

$$\chi(t, x) = \frac{2t}{\pi X}\left[\frac{I_0}{\delta^2}(E/F-x^2)+\frac{I_1}{\delta^4}\{(t^2+\tfrac{3}{2}-\tfrac{3}{2}a^2)(E/F-x^2) + 3x^2(1-x^2)-E/F+2a^2E/F-a^2\}+0(\delta^{-6})\right] \quad (8.16)$$

$$\Gamma(t, x) = \frac{2t^2}{\pi X}\left[\frac{J_0}{\delta^3}(E/F-x^2)+\frac{J_1}{\delta^5}\{(t^2+3-3a^2)(E/F-x^2) + 6x^2(1-x^2)-2E/F+4a^2E/F-2a^2\}+0(\delta^{-7})\right] \quad (8.17)$$

$$\chi_1(t, x) = \frac{2t^2}{\pi X}\left[\frac{K_0x^2}{\delta^4}\{(1+a^2)/2-x^2\}+0(\delta^{-6})\right] \quad (8.18)$$

$$\Gamma_1(t, x) = \frac{2t}{\pi X}\left[\frac{L_0x^2}{\delta^3}\{(1+a^2)/2-x^2\}+\frac{L_1x^2}{\delta^5}\left\{(x^2+t^2)\left(\frac{1+a^2}{2}-x^2\right) + (1-a^2)^2/8\right\}+0(\delta^{-7})\right] \quad (8.19)$$

where

$I_0 = \int_0^\infty uI(u)\,du$, $I_1 = -\frac{1}{3\,!}\int_0^\infty u^3I(u)\,du$, $J_0 = \frac{1}{2}\int_0^\infty u^2J(u)\,du$

$J_1 = -\frac{1}{4\,!}\int_0^\infty u^4J(u)\,du$, $K_0 = \frac{1}{2}\int_0^\infty u^3K(u)\,du$, $L_0 = \int_0^\infty u^2L(u)\,du$, $L_1 = -\frac{1}{3\,!}$ $\times\int_0^\infty u^4L(u)\,du$.

The values of these constants for the two problems under study are:

Problem 1

$$I_0 = -3/2, \quad I_1 = 15/16, \; J_0 = -3/4, \quad J_1 = 15/32$$
$$K_0 = -21/16, \quad L_0 = -3/2, \quad L_1 = 15/8$$

Problem 2

$I_0 = (2b+5)/4(3-4\eta)$, $I_1 = -(b+7)/8(3-4\eta)$, $J_0 = (c+3)/4(3-4\eta)$,
$J_1 = -(2c+5)/32(3-4\eta)$, $K_0 = 3(b+3)/8(3-4\eta)$,
$L_0 = (c+3)/2(3-4\eta)$, $L_1 = -(2c+15)/32(3-4\eta)$.

Suppose that the solutions of (8.14) and (8.15) can be written in the form

$$h(x^2) = h_0(x^2)+\frac{h_1(x^2)}{\delta}+\frac{h_2(x^2)}{\delta^2}+\ldots+\frac{h_5(x^2)}{\delta^5}+0(\delta^{-6}) \quad (8.20)$$

$$g(x^2) = g_0(x^2)+\frac{g_1(x^2)}{\delta}+\frac{g_2(x^2)}{\delta^2}+\ldots+\frac{g_5(x^2)}{\delta^5}+0(\delta^{-6}). \quad (8.21)$$

Substituting these values of $h(x^2)$ and $g(x^2)$ in (8.14) and (8.15) we get

$$h_0(x^2) = \phi_1(x),\ h_1(x^2) = 0$$

$$h_2(x^2) = \frac{2I_0}{\pi X}(E/F-x^2)\int_a^1 th_0(t^2)\,dt$$

$$h_3(x^2) = \frac{2J_0}{\pi X}(E/F-x^2)\int_a^1 tg_0(t^2)\,dt$$

$$h_4(x^2) = \frac{2}{\pi X}\int_a^1 [I_0(E/F-x^2)h_2(t^2)+I_1\{(t^2+3/2-3a^2/2)(E/F-x^2)$$
$$+3x^2(1-x^2)-E/F+2a^2E/F-a^2\}h_0(t^2)+J_0(E/F-x^2)g_0(t^2)]t\,dt$$

$$h_5(x^2) = \frac{2}{\pi X}\int_a^1 [I_0(E/F-x^2)h_3(t^2)+J_1\{6x^2(1-x^2)$$
$$+(t^2+3-3a^2)(E/F-x^2)-2E/F+4a^2E/F-2a^2\}g_0(t^2)]t\,dt$$

$$g_0(x^2) = \psi_1(x),\ g_1(x^2) = 0,\quad g_2(x^2) = 0$$

$$g_3(x) = \frac{2L_0}{\pi X}x^2\left(\frac{1+a^2}{2}-x^2\right)\int_a^1 th_0(t^2)\,dt$$

$$g_4(x^2) = \frac{2K_0}{\pi X}x^2\left(\frac{1+a^2}{2}-x^2\right)\int_a^1 tg_0(t^2)\,dt$$

$$g_5(x^2) = \frac{2}{\pi X}\int_a^1 \left[L_0x^2\left(\frac{1+a^2}{2}-x^2\right)h_2(t^2)+L_1x^2\left\{(x^2+t^2)\left(\frac{1+a^2}{2}-x^2\right)\right.\right.$$
$$\left.\left.+(1-a^2)^2/8\right\}h_0(t^2)\right]t\,dt.$$

The method given here can very well be applied to any system of simultaneous triple integral equations involving trigonometric kernels. The technique for solving simultaneous triple integral equations involving Bessel functions which are likely to occur in three-part boundary value problems of solids will be published elsewhere.

We shall study the case when the cracks are opened by constant normal internal pressure and the shearing stress on the surface of cracks is zero. So that

$p(y) = \text{constant} = \frac{\pi}{2}p$ and $\tau(y) = 0$. Hence $P(y) = (2\pi)^{\frac{1}{2}}(1-\eta)p/\mu = P$, and $T(y) = 0$. This gives

$$g_0(x^2) = g_1(x^2) = g_2(x^2) = g_4(x^2) = h_1(x^2) = h_3(x^2) = h_5(x^2) \doteq 0$$
$$h_0(x^2) = P(E/F-x^2)X$$

$$h_2(x^2) = -I_0 P\alpha(E/F-x^2)/2X$$

$$h_4(x^2) = \frac{P\alpha}{4X}\left[I_0^2\,\alpha(E/F-x^2)+2I_1(x^4+\beta x^2+\gamma)\right]$$

$$g_3(x^2) = \frac{PL_0\alpha}{4X}\,x^2(2x^2-a^2-1)$$

$$g_5(x^2) = \frac{P\alpha}{8X}\left[L_0I_0\alpha x^2(a^2+1-2x^2)+4L_1x^2(x^4+\lambda x^2+\theta)\right]$$

where

$$\alpha = 1+a^2-2E/F,\quad \beta = \frac{(1-a^2)^2}{4}-a^2-1,\quad \gamma = a^2+\beta E/F$$

$$\lambda = \frac{(1-a^2)^2}{4\alpha},\quad \theta = -\tfrac{1}{8}\left\{(1-a^2)^2\left(1+\frac{1+a^2}{\alpha}\right)+2(1+a^2)^2\right\}$$

9. The Stress Intensity Factor

It is of interest to workers in fracture mechanics to calculate the stress intensity factors at the ends of the cracks. They are given by the relations

$$\chi_a = \lim_{x\to a-}\,(a-x)^{\frac{1}{2}}[\sigma_y(x,0)]_{0<x<a} \quad \ldots \quad (9.1)$$

$$\chi = \lim_{x\to 1+}\,(x-1)^{\frac{1}{2}}[\sigma_y(x,0)]_{x>1} \quad \ldots \quad (9.2)$$

$$\zeta_a = \lim_{x\to a-}\,(a-x)^{\frac{1}{2}}[\tau_{xy}(x,0)]_{0<x<a} \quad \ldots \quad (9.3)$$

$$\zeta = \lim_{x\to 1+}\,(x-1)^{\frac{1}{2}}[\tau_{xy}(x,0)]_{x>1} \quad \ldots \quad (9.4)$$

We have

$$\frac{(2\pi)^{\frac{1}{2}}(1-\eta)\sigma_y(x,0)}{\mu} = \frac{d}{dx}\int_0^\infty N(\xi)\sin\xi x\,d\xi+\int_0^\infty[I(\xi\delta)N(\xi)+J(\xi\delta)M(\xi)]\xi\cos\xi x\,d\xi.$$

Substituting the values of $M(\xi)$ and $N(\xi)$ from (8.5) and (8.6) we get

$$\frac{(2\pi)^{\frac{1}{2}}(1-\eta)\sigma_y(x,0)}{\mu} = \int_a^1\frac{th(t^2)}{t^2-x^2}dt+\int_a^1 h(t^2)\int_0^\infty I(\xi\delta)\sin\xi t\cos\xi x\,d\xi\,dt$$

$$+\int_a^1\frac{g(t^2)}{t}\int_0^\infty J(\xi\delta)\cos\xi x(1-\cos\xi t)d\xi\,dt. \quad \ldots \quad (9.5)$$

Similarly

$$\frac{(2\pi)^{\frac{1}{2}}(1-\eta)\tau_{xy}(x,0)}{\mu} = -\frac{1}{x}\int_a^1\frac{tg(t^2)}{t^2-x^2}dt-\int_a^1\frac{g(t^2)}{t}\int_0^\infty K(\xi\delta)\sin\xi x(1-\cos\xi t)d\xi\,dt$$

$$-\int_a^1 h(t^2)\int_0^\infty L(\xi\delta)\sin\xi x\sin\xi t\,d\xi\,dt. \quad \ldots \quad (9.6)$$

Substituting the values of $h(t^2)$ and $g(t^2)$ and evaluating the integrals with the help of the results given in section 7, we get

$$[\sigma_y(x,0)]_{0<x<a} = p\Bigg[\left(\frac{E/F-x^2}{\{(a^2-x^2)(1-x^2)\}^{\frac{1}{2}}}-1\right)\left(1-\frac{I_0\alpha}{2\delta^2}+\frac{I_0^2\alpha^2}{4\delta^4}\right)$$
$$+\frac{2I_1}{\delta^4}\left(\frac{3x^4+\beta x^2+\gamma}{\{(a^2-x^2)(1-x^2)\}^{\frac{1}{2}}}+3x^2+\tfrac{3}{2}(1+a^2)+\beta\right)+\frac{I_0\alpha}{2\delta^2}-\frac{I_0^2\alpha^2}{4\delta^4}+\frac{I_1\alpha}{2\delta^4}(3x^4+a^2+\beta+1)$$
$$+0(\delta^{-6})\Bigg] \quad .. \quad (9.7)$$

$$[\sigma_y(x,0)]_{x>1} = p\Bigg[\left(\frac{x^2-E/F}{\{(x^2-a^2)(x^2-1)\}^{\frac{1}{2}}}-1\right)\left(1-\frac{I_0\alpha}{2\delta^2}+\frac{I_0^2\alpha^2}{4\delta^4}\right)$$
$$+\frac{I_0\alpha}{2\delta^2}-\frac{I_0^2\alpha^2}{4\delta^4}-\frac{2I_1}{\delta^4}\left\{\frac{3x^4+\beta x^2+\gamma}{\{(x^2-a^2)(x^2-1)\}^{\frac{1}{2}}}-3x^2-\tfrac{3}{2}(1+a^2)-\beta\right\}$$
$$+\frac{I_1\alpha}{2\delta^4}(3x^4+a^2+\beta+1)+0(\delta^{-6})\Bigg] \quad .. \quad .. \quad .. \quad .. \quad .. \quad (9.8)$$

$$[\tau_{xy}(x,0)]_{0<x<a} = \frac{p\alpha}{4}\Bigg[x\left(\frac{1+a^2-2x^2}{\{(a^2-x^2)(1-x^2)\}^{\frac{1}{2}}}-1\right)\left(\frac{L_0}{\delta^3}-\frac{\alpha L_0 I_0}{2\delta^5}\right)+\frac{L_0}{\delta^3}$$
$$+\frac{L_1}{2\delta^5}(2x^2+2\lambda+a^2+1)-\frac{L_1I_0\alpha}{2\delta^5}-\frac{L_1x}{\delta^5}\left(\frac{2(x^4+\lambda x^2+\theta)}{\{(a^2-x^2)(1-x^2)\}^{\frac{1}{2}}}+2(x^2+\lambda)+a^2+1\right)$$
$$+0(\delta^{-7})\Bigg] \quad .. \quad (9.9)$$

$$[\tau_{xy}(x,0)]_{x>1} = \frac{p\alpha}{4}\Bigg[x\left(\frac{2x^2-a^2-1}{\{(x^2-a^2)(x^2-1)\}^{\frac{1}{2}}}-1\right)\left(\frac{L_0}{\delta^3}-\frac{\alpha L_0 I_0}{2\delta^5}\right)+\frac{L_0}{\delta^3}$$
$$-\frac{L_1I_0\alpha}{2\delta^5}+\frac{L_1}{2\delta^5}(2x^2+2\lambda+a^2+1)+\frac{L_1x}{\delta^5}\left(\frac{2(x^4+\lambda x^2+\theta)}{\{(x^2-a^2)(x^2-1)\}^{\frac{1}{2}}}-2(x^2+\lambda)-a^2-1\right)$$
$$+0(\delta^{-7})\Bigg]. \quad .. \quad (9.10)$$

Hence the stress intensity factors are

$$\chi_a = \frac{p}{\{2a(1-a^2)\}^{\frac{1}{2}}}\left[(E/F-a^2)\left(1-\frac{I_0\alpha}{2\delta^2}+\frac{I_0^2\alpha^2}{4\delta^4}\right)+\frac{2I_1}{\delta^4}(3a^4+\beta a^2+\gamma)+0(\delta^{-6})\right] \quad .. \quad (9.11)$$

$$\chi = \frac{p}{\{2(1-a^2)\}^{\frac{1}{2}}}\left[(1-E/F)\left(1-\frac{I_0\alpha}{2\delta^2}+\frac{I_0^2\alpha^2}{4\delta^4}\right)-\frac{2I_1}{\delta^4}(3+\beta+\gamma)+0(\delta^{-6})\right] \quad (9.12)$$

$$\zeta_a = \frac{\alpha pa}{4\{2a(1-a^2)\}^{\frac{1}{2}}}\left[(1-a^2)\left(\frac{L_0}{\delta^3}-\frac{L_0I_0\alpha}{2\delta^5}\right)-\frac{2L_1}{\delta^5}(a^4+\lambda a^2+\theta)+0(\delta^{-7})\right] \quad (9.13)$$

$$\zeta = \frac{p\alpha}{4\{2(1-a^2)\}^{\frac{1}{2}}}\left[(1-a^2)\left(\frac{L_0}{\delta^3}-\frac{L_0I_0}{2\delta^5}\right)+\frac{2L_1}{\delta^5}(1+\lambda+\theta)+0(\delta^{-7})\right]. \quad (9.14)$$

When $a \to 0$, the two cracks merge into a single crack of two units. Then $E \to 1$, $F \to \infty$, $\lambda = 1/4$, $\theta = -1/2$, $\alpha = 1$, $\beta = -1$, $\gamma = 0$. Hence we have

$$\chi = \frac{p}{\sqrt{2}}\left[1 - \frac{I_0}{2\delta^2} + \frac{I_0^2}{4\delta^4} - \frac{4I_1}{\delta^4} + 0(\delta^{-6})\right] \quad \ldots \quad (9.15)$$

$$\zeta = \frac{p}{4\sqrt{2}}\left[\frac{L_0}{\delta^3} - \frac{L_0 I_0}{2\delta^5} + \frac{3L_1}{2\delta^5} + 0(\delta^{-7})\right]. \quad \ldots \quad (9.16)$$

10. Shape of the Crack

We shall consider only the case when free boundary is stress free. The other case can be dealt in like manner. With the help of the relations given in section 6 it is easy to show that the components of displacement can be expressed as

$$u(x, 0) = \frac{1}{(2\pi)^{\frac{1}{2}}(1-\eta)}\int_0^\infty \Big[M(\xi)\big((1-\eta) + \{(1-\eta)(1-2\xi\delta) + \xi^2\delta^2\}e^{-2\xi\delta}\big)$$
$$+ N(\xi)\big((\tfrac{1}{2}-\eta) + \{(\tfrac{1}{2}-\eta)(1-2\xi\delta) + \xi^2\delta^2\}e^{-2\xi\delta}\big)\Big] \sin \xi x \, d\xi \quad \ldots \quad (10.1)$$

$$v(x, 0) = \frac{1}{(2\pi)^{\frac{1}{2}}(1-\eta)}\int_0^\infty \Big[M(\xi)\big((\tfrac{1}{2}-\eta) + \{\xi^2\delta^2 - (\tfrac{1}{2}-\eta)(1-2\xi\delta)\}e^{-2\xi\delta}\big)$$
$$+ N(\xi)\big((1-\eta) + \{(1-\eta)(1+2\xi\delta) + \xi^2\delta^2\}e^{-2\xi\delta}\big)\Big] \cos \xi x \, d\xi. \quad \ldots \quad (10.2)$$

Substituting the values of $M(\xi)$ and $N(\xi)$ from (8.5) and (8.6) we get for $a < x < 1$

$$(2\pi)^{\frac{1}{2}}(1-\eta)u(x, 0) = \frac{\pi}{2}(1-\eta)\int_a^x \frac{g(t^2)}{t}\,dt + \frac{1-2\eta}{4}\int_a^1 h(t^2)\log\left|\frac{t+x}{t-x}\right| dt$$
$$+ \int_a^1 \frac{g(t^2)}{t}\,dt \int_0^\infty \{\xi^2\delta^2 + (1-\eta)(1-2\xi\delta)\}e^{-2\xi\delta} \sin \xi x(1-\cos \xi t)\frac{d\xi}{\xi}\,dt$$
$$+ \int_a^1 h(t^2)\int_0^\infty \{\xi^2\delta^2 + (\tfrac{1}{2}-\eta)(1-2\xi\delta)\}e^{-2\xi\delta} \sin \xi x \sin \xi t \frac{d\xi}{\xi}\,dt \quad \ldots \quad (10.3)$$

$$(2\pi)^{\frac{1}{2}}(1-\eta)v(x, 0) = \frac{\pi}{2}(1-\eta)\int_x^1 h(t^2)\,dt + \frac{1-2\eta}{4}\int_a^1 \frac{g(t^2)}{t}\log\left|1 - \frac{t^2}{y^2}\right| dt$$
$$+ \int_a^1 \frac{g(t^2)}{t}\int_0^\infty \{\xi^2\delta^2 - (\tfrac{1}{2}-\eta)(1-2\xi\delta)\}e^{-2\xi\delta} \cos \xi x(1-\cos \xi t)\frac{d\xi}{\xi}\,dt$$
$$+ \int_a^1 h(t^2)\int_0^\infty \{\xi^2\delta^2 + (1-\eta)(1+2\xi\delta)\}e^{-2\xi\delta} \cos \xi x \sin \xi t \frac{d\xi}{\xi}\,dt \quad \ldots \quad (10.4)$$

Substituting the value of $g(t^2)$ and $h(t^2)$, after some calculations carried out with the help of results given in section 7 we get

$$[u(x, 0)]_{a<x<1} = \frac{\pi p}{4\mu}\left[(1-2\eta)\left(\frac{1-I_0\alpha}{2\delta^2}\right)\left(\frac{\pi}{2F}-x\right)-\frac{\alpha x(1+\eta)}{4\delta^2}\right.$$

$$\left.+\frac{L_0\alpha(1-\eta)}{2\delta^3}\{(1-x^2)(x^2-a^2)\}^{\frac{1}{2}}+0(\delta^{-4})\right] \quad \ldots \quad \ldots \quad (10.5)$$

$$[v(x, 0)]_{a<x<1} = \frac{\pi p}{16\mu}\left[D(\phi)\left\{1-\frac{I_0\alpha}{2\delta^2}+\frac{I_0^2\alpha^2+2I_1\alpha\beta}{4\delta^4}\right\}8(1-\eta)\right.$$

$$-\frac{I_1}{2\delta^4}(1-\eta)\{(1-a^2)^2E[(1-a^2)^{\frac{1}{2}}, \phi]+2\alpha x((1-x^2)(x^2-a^2))^{\frac{1}{2}}\}$$

$$-\frac{\alpha}{\delta}(5-4\eta)\left(1-\frac{I_0\alpha}{\delta^2}\right)+\frac{(7-4\eta)}{6\delta^3}\alpha\left\{3x^2+\frac{a^2+1}{2}+\frac{(1-a^2)^2}{4\alpha}\right\}$$

$$\left.+\frac{L_0}{4\delta^3}(1-2\eta)(1+a^2-2x^2)+0(\delta^{-5})\right] \quad \ldots \quad \ldots \quad \ldots \quad (10.6)$$

where

$$\sin\phi = \{(1-x^2)/(1-a^2)\}^{\frac{1}{2}},\ D(\phi) = \frac{E}{F}F[(1-a^2)^{\frac{1}{2}}, \phi]-E[(1-a^2)^{\frac{1}{2}}, \phi].$$

For a single crack the components of displacement vector are

$$[u(x, 0)]_{0<x<1} = \frac{\pi p x}{4\mu}\left[(2\eta-1)\left(1-\frac{I_0}{2\delta^2}\right)-\frac{1+\eta}{2\delta^2}+\frac{L_0}{2\delta^3}(1-\eta)(1-x^2)^{\frac{1}{2}}+0(\delta^{-4})\right]$$

$$\ldots \quad (10.7)$$

$$[v(x, 0)]_{0<x<1} = \frac{\pi p}{16\mu}\left[(1-\eta)(1-x^2)^{\frac{1}{2}}\left\{8-\frac{4I_0}{\delta^2}+\frac{2(I_0^2-2I_1)}{\delta^4}-\frac{I_1x^2}{\delta^4}\right\}\right.$$

$$\left.-(5-4\eta)\left(\frac{1}{\delta}-\frac{I_0}{\delta^3}\right)+\frac{7-4\eta}{2}(x^2+\tfrac{1}{4})+\frac{L_0}{4\delta^3}(1-2x^2)(1-2\eta)+0(\delta^{-5})\right]$$

$$\ldots \quad (10.8)$$

11. Crack Energy

The energy required to open the crack is given by

$$W = \int_a^1 p\, v(x, 0)\, dx. \quad \ldots \quad \ldots \quad \ldots \quad (11.1)$$

Substituting the value of $v(x, 0)$ from (10.4), after some calculations we get

$$W = \frac{\pi^2p^2}{16\mu}\left[(1-\eta)\left\{\alpha-\frac{I_0\alpha^2}{2\delta^2}+\frac{I_0^2\alpha^3}{4\delta^4}-\frac{I_1}{4\delta^4}(1+a^2)\left(2\alpha+\frac{1-a^2}{\alpha}\right)\right\}\right.$$

$$+\alpha(1-a)\left\{(5-4\eta)\left(\frac{1}{\delta}-\frac{I_0}{\delta^3}\right)+\frac{7-4\eta}{6\delta^3}\left(\frac{3+2a+3a^2}{2}+\frac{(1-a^2)^2}{4\alpha}\right)\right.$$

$$\left.\left.+\frac{L_0(1-2\eta)(1-a)^2}{12\delta^3}\right\}+0(\delta^{-5})\right]. \quad \ldots \quad (11.2)$$

For a single crack the energy required to open it is obtained from the above expression by tending $a \to 0$. Hence we have

$$2W = \frac{\pi^2 p^2}{8\mu}\left[(1-\eta)\left(1-\frac{I_0}{2\delta^2}+\frac{I_0^2}{4\delta^4}-\frac{3I_1}{4\delta^4}\right)-(5-4\eta)\left(\frac{1}{\delta}-\frac{I_0}{\delta^3}\right)+\frac{7(7-4\eta)}{24\delta^3}\right.$$
$$\left.+\frac{L_0(1-2\eta)}{12\delta^3}+0(\delta^{-5})\right] \quad .. \quad (11.3)$$

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Added in proof: The iterative procedure adopted to solve the integral equations needs some justification. We have called equations (8.14) and (8.15) simultaneous Fredholm integral equations, although their kernels $X(t, x)$, $\Gamma(t, x)$, etc., do not belong to L^2 class—the type of kernels which as a rule is called regular. In spite of this, all fundamental Fredholm theorems are applicable to these equations, if they are stated in a suitable manner. This will be shown by reduction of (8.14) and (8.15) to simultaneous Fredholm equations with bounded kernels which are L^2. Multiply (8.14) and (8.15) by X and put

$$H(x^2) = X \,.\, h(x^2) \ \text{ and } \ G(x^2) = X \,.\, g(x^2)$$

and replace the variable t by a new variable θ by letting

$$t^2 = a^2 \cos^2 \theta + \sin^2 \theta.$$

After some manipulations we get

$$H(x^2) = X\phi_1(x)+\int_0^{\pi/2} H(t^2)n(\theta, x)\, d\theta+\int_0^{\pi/2} G(t^2)m(\theta, x)\, d\theta$$

$$G(x^2) = X\psi_1(x)+\int_0^{\pi/2} G(t^2)m'(\theta, x)\, d\theta+\int_0^{\pi/2} H(t^2)n'(\theta, x)\, d\theta.$$

From (8.16) we derive that

$$n(\theta, x) = \left[\frac{I_0}{\delta^2}(E/F-x^2)+\frac{I_1}{\delta^4}\left\{\left(a^2\cos^2\theta+\sin^2\theta-\frac{3}{2}a^2+\frac{3}{2}\right)(E/F-x^2)\right.\right.$$
$$\left.\left.+3x^2(1-x^2)-E/F+2a^2E/F-a^2\right\}\right]$$

and similar expressions for $m(\theta, x)$, etc., can be obtained. The above integral equations have kernels which are bounded and L^2 $(0, \pi/2)$.