UNION CURVES IN CONTACT MANIFOLDS

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Union curves in Riemannian spaces were studied by Springer, Upadhyay and others and in Finsler spaces by Prakash, Mishra and others. The purpose of the present paper is to obtain differential equation of union curves in almost contact affine manifolds.

Introduction

A curve in an ordinary Euclidean surface is a union curve relative to a congruence if its osculating plane at each point contains the line of that rectilinear congruence through that point. Union curves in Riemannian spaces were studied by Springer (1945, 1950), Upadhyay (1965) and others and in Finsler spaces by Prakash and Behari (1961), Mishra and Sinha (1965) and others. Springer (1945) obtained the differential equation of union curves on a metric surface in Euclidean spaces. He (1950) obtained the differential equation of union curves on a hypersurface V_n immersed in a Riemannian manifold V_{n+1}. In this paper, differential equation of union curves has been obtained in almost contact affine manifolds.

Hypersurfaces in Almost Contact Spaces

Let M_{2n+1} be a differentiable affine manifold of differentiability of class C^∞, the affine connection L^i_{jk} being given by

$$\frac{\partial h^i_{(\alpha)}}{\partial x^j}+h^k_{(\alpha)}L^i_{jk}=0$$

where $h^k_{(\alpha)}$ (Thomas 1934) are $(2n+1)$ fundamental parallel contravariant vector fields in M_{2n+1}. Let M_{2n+1} be equipped with an almost contact structure tensor field (f^i_j, f_i, f^i) satisfying

$$\left.\begin{aligned} f^k_i f^i_j - f_j f^k &= -\delta^k_j \\ f^i_j f^j = 0, \quad f^i_j f_i = 0, &\quad f^i f_i = 1 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

and we denote $\{M_{2n+1}, (f^i_j, f_i, f^j)\}$ by M_{2n+1} and call it an almost contact affine manifold.

Now consider a $2n$-dimensional hypersurface K_{2n} of differentiability of class C^{∞} in M_{2n+1}. Let the hypersurface K_{2n} be expressed by $x^i = x^i(y^a)$, wherein x^i are a system of local coordinates in M_{2n+1} and y^a that in K_{2n}. We adopt the convention that the indices $h, i, j, k, \ldots$ run over the range $1, 2, \ldots, (2n+1)$ and the indices $a, b, c, \ldots$ over the range $1, 2, \ldots 2n$ unless otherwise stated.

Let $B^i_a = \partial_a x^i$, $\partial_a = \dfrac{\partial}{\partial y^a}$, so that B^i_a are $2n$ local vector fields along K_{2n} and linearly independent at each point of K_{2n}. Let C^i be the contravariant component of a vector field normal to K_{2n}. Hence we have

$$B^i_a B^b_i = \delta^b_a, \quad B^i_a C_i = 0, \quad B^a_i C^i = 0$$

$$B^i_a B^a_j + C^i C_j = \delta^i_j$$

where (B^a_i, C^i) is the inverse matrix of $\begin{pmatrix} B^i_a \\ C_i \end{pmatrix}$.

The induced affine connection L^a_{bc} of K_{2n} will be given by

$$L^a_{cb} = (\partial_c B^h_b + B^j_c B^i_b L^h_{ji}) B^a_h.$$

The Van der Waerden Borlotti covariant derivatives of B^i_a are given by

$$\nabla_c B^h_b = H_{cb} C^h$$

where H_{cb} is the second fundamental tensor of the hypersurface K_{2n}.

The condition for the distribution $(B^i_a, C^i = f^i)$, which is determined by $f_i dx^i = 0$, to be completely integrable is

$$B^k_c B^j_b (\nabla_k f_j - \nabla_j f_k) = 0$$

$$\text{or} \quad B^j_b \nabla_c f_j - B^k_c \nabla_b f_k = 0$$

where f_k is the structure vector field of M_{2n+1}.

It is known that (Eum 1968) if the distribution $(B^i_a, C^i = f^i)$ determined by $f_i dx^i = 0$ in $\{M_{2n+1}, (f^i_j, f_i, f^j)\}$ is completely integrable, then M_{2n+1} includes an almost complex hypersurface $\{K_{2n}, \phi^a_b\}$ whose normal vector is f^i and whose structure tensor ϕ^a_b satisfies

$$B^j_a f^i_j = B^i_b \phi^b_a$$

$$\phi^b_c \phi^a_b = -\delta^a_c.$$

Vector Fields in K_{2n}

Let u^i be the contravariant component in x's of a vector field in K_{2n} and ξ^a be the corresponding contravariant component in the y's, then

$$u^i = B^i_a \xi^a. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The covariant derivative of eqn. (1) with respect to y^b yields

$$\nabla_b u^i = (\nabla_b B^i_a)\xi^a + B^i_a(\nabla_b \xi^a). \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Let C be any curve in K_{2n} with equation $y^a = y^a(t)$ (t being a parameter) and let p^i be the contravariant component in x's of the derived vector relative to M_{2n+1} of a vector of the field along C and q^a the corresponding contravariant components in y's of the derived vector relative to K_{2n} of the same vector along C, then multiplying eqn. (2) by $\frac{dy^b}{dt}$ we get

$$p^i = H_{ba}\,\xi^a \frac{dy^b}{dt} f^i + B^i_a q^a. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Now let u^i be the contravariant component in x's of a vector in the direction of a curve of a congruence of curves, one curve of which passes through each point of K_{2n} and let u_i be the corresponding covariant component of the vector such that

$$u^i u_i = 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The vector with components u^i will in general be given by

$$u^i = B^i_a s^a + r f^i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where in view of eqn. (4) s^a and r are connected by the relation

$$s^a s_a = 1 - r^2.$$

Substituting the value of f^i from eqn. (5) in eqn. (A), we get the relation which the vector u^i satisfies in relation to the structure tensor in contact space.

Union Curves of the Hypersurface

We call a curve C in K_{2n} a union curve if u^i is tangential to the variety determined by the tangent to C and the first curvature vector of C regarded as a curve of M_{2n+1}.

Thus u^i will be a linear combination of $B^i_a \xi^a$ and p^i so that we get

$$s^a B^i_a + r f^i = l\, B^i_a \xi^a + m p^i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where l and m are to be determined. The ξ^a in eqn. (3) are now given by $\frac{dy^a}{dt}$ so that

$$p^i = H_{ba} \frac{dy^a}{dt} \frac{dy^b}{dt} f^i + B^i_a q^a. \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

If we write $k_n = H_{ba} \frac{by^a}{dt} \frac{dy^b}{dt}$, then eqn. (7) can be written as

$$s^a B^i_a + r f^i = l\, B^i_a \frac{dy^a}{dt} + m(k_n f^i + B^i_a q^a). \quad \ldots \quad \ldots \quad (8)$$

On multiplying eqn. (8) by f_i and B_i^b, we get

$$r = mk_n \text{ giving } m = \frac{1}{k_n} r$$

and

$$s^b = l \frac{dy^b}{dt} + mq^b. \qquad \qquad (9)$$

Now multiplying eqn. (9) by $G_{ab} \frac{dy^a}{dt}$, we obtain

$$G_{ab} \frac{dy^a}{dt} s^b = l\, G_{ab} \frac{dy^a}{dt} \frac{dy^b}{dt}$$

$$\text{or} \quad l = \frac{1}{C} G_{ab} \frac{dy^a}{dt} s^b$$

where

$$C = G_{ab} \frac{dy^a}{dt} \frac{dy^b}{dt}.$$

Here G_{ab} is the affine tensor (Indrani, *in press*) which is used to raise and lower the indices. Thus eqn. (9) reduces to

$$s^a = \frac{1}{C} G_{db} \frac{dy^d}{dt} \frac{dy^a}{dt} s^b + \frac{1}{k_n} rq^a$$

or

$$q^a - k_n \left[\frac{s^a}{r} - \frac{1}{C} \frac{s^b}{r} G_{db} \frac{dy^d}{dt} \frac{dy^a}{dt} \right] = 0. \qquad \qquad (10)$$

Equation (10) now gives the differential equation of the union curve in the hypersurface K_{2n}.

Subspaces of Almost Contact Spaces

Consider a $2m$-dimensional subspace K_{2m} of class C^∞ of M_{2n+1} covered by a system of coordinate neighbourhoods (y^λ) where λ, μ, ϑ ... run over the range $1, 2, \ldots m, \bar{1}, \bar{2}, \ldots \bar{m}$, and $a, b, c, \ldots$ run over $1, 2, \ldots m$. Introduce in each coordinate neighbourhood (y^λ) complex coordinates (z^a) defined by

$$z^a = y^a + \sqrt{-1} y^{\bar{a}}.$$

We cover K_{2m} by a system of complex coordinate neighbourhood (z^a) in such a way that in the intersection of two complex neighbourhoods (z^a) and $(z^{a'})$ we have

$$z^{a'} = \phi^{a'}(z^a), \quad \left| A_a^{a'} \right| \neq 0$$

where $\phi^{a'}(z^a)$ are regular complex functions of complex variables $z^1, z^2 \ldots z^m$ and $\left| A_a^{a'} \right| = \partial_a z^{a'}$. K_{2m} covered by complex coordinate neighbourhoods admits a complex structure and is, therefore, a complex manifold (Yano 1965). We now study the union curves in this complex sub-manifold of M_{2n+1}.

Let B^i_λ be $2m$ local vector fields along K_{2m} and linearly independent at each point of K_{2m}. Let $N^i_{(\alpha)}$ $(\alpha = 2m+1, \ldots (2n+1))$ be the contravariant component in the coordinate system x^i of a system of normals to M_{2n+1}. We have

$$N^i_{(\alpha)} N_{(\beta)i} = \delta^\alpha_\beta, \quad N^i_{(\alpha)} B^\lambda_i = 0.$$

$$B^i_\lambda N_{(\alpha)i} = 0.$$

If the distribution $\left(B^i_\lambda, f^i = \underset{\alpha}{\Sigma} a_\alpha N^i_{(\alpha)}\right)$ determined by $f_i dx^i = 0$ in $\{M_{2n+1}, (f^i_j, f_i, f^j)\}$ is completely integrable, then M_{2n+1} includes a complex subspace K_{2m} whose normal vectors are $N^i_{(\alpha)}$.

Now the covariant derivative of B^i_λ along the subspace will be given by $\nabla_\mu B^i_\lambda = H_{\lambda\mu} f^i$, where $H_{\lambda\mu}$ are the components of a symmetric covariant tensor of second order.

Vector Fields in K_{2m}

The contravariant components q^i and p^λ of the first curvature vector with respect to M_{2n+1} and K_{2m} are related by

$$q^i = H_{\lambda\mu} \frac{dy^\lambda}{dt} \frac{dy^\mu}{dt} f^i + B^i_\lambda p^\lambda. \qquad \ldots \quad \ldots \quad \ldots \quad (11)$$

There are $(2n+1-2m)$ congruences specified by vectors $\eta_{(\alpha)}$ whose components along the tangent and normals $N^i_{(\alpha)}$ are given by

$$\eta^i_{(\alpha)} = t^\lambda_{(\alpha)} B^i_\lambda + C_{(\alpha)} f^i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where t^λ and $C_{(\alpha)}$ are related by

$$t^\lambda_{(\alpha)} t_{(\alpha)\lambda} = 1 - (C_{(\alpha)})^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Union Curves in Subspace

A curve C in the subspace is said to be a union curve relative to a set of $(2n+1-2m)$ congruences $\eta_{(\alpha)}$ if $\eta_{(\alpha)}$ are tangential to the variety determined by the tangent to C and the first curvature of C with respect to K_{2m}.

Let the vector $\eta_{(\alpha)}$ with components $\eta^i_{(\alpha)}$ be in the direction of the congruence. It can be expressed as a linear combination of $B^i_\lambda \frac{dy^\lambda}{dt}$ and the first curvature vector q^i of the union curve C.

$$\eta^i_{(\alpha)} = u_{(\alpha)} \frac{dy^\lambda}{dt} B^i_\lambda + v_{(\alpha)} q^i \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where $u_{(\alpha)}$ and $v_{(\alpha)}$ are to be determined. From eqns. (12) and (14) we have

$$t^\lambda_{(\alpha)} B^i_\lambda + C_{(\alpha)} f^i = u_{(\alpha)} B^i_\lambda \frac{dy^\lambda}{dt} + v_{(\alpha)} \left[H_{\lambda\mu} \frac{dy^\lambda}{dt} \frac{dy^\mu}{dt} f^i + B^i_\lambda p^\lambda \right]. \qquad \ldots \quad (15)$$

Multiplying eqn. (15) by f_ι and B^μ_ι, we get

$$C_{(\alpha)} = v_{(\alpha)} H_{\lambda\mu} \frac{dy^\lambda}{dt} \frac{dy^\mu}{dt}$$

$$\text{giving} \quad v_{(\alpha)} = \frac{1}{K'} C_{(\alpha)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$\left(K' = H_{\lambda\mu} \frac{dy^\lambda}{dt} \frac{dy^\mu}{dt} \right)$$

$$\text{and} \quad t^\lambda_{(\alpha)} = u_{(\alpha)} \frac{dy^\lambda}{dt} + v_{(\alpha)} p^\lambda. \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

Again, multiplying eqn. (17) by $a_{\lambda\mu} \dfrac{dy^\mu}{dt}$, we get

$$a_{\lambda\mu} t^\lambda_{(\alpha)} \frac{dy^\mu}{dt} = u_{(\alpha)} a_{\lambda\mu} \frac{dy^\lambda}{dt} \frac{dy^\mu}{dt}$$

$$\text{or} \quad u_{(\alpha)} = \frac{1}{C'} a_{\lambda\mu} \frac{dy^\mu}{dt} t^\lambda_{(\alpha)} \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$\text{where} \quad C' = a_{\lambda\mu} \frac{dy^\lambda}{dt} \frac{dy^\mu}{dt}.$$

Thus substituting for $u_{(\alpha)}$ and $v_{(\alpha)}$ from eqns. (16) and (18) in eqn. (17), we get

$$t^\lambda_{(\alpha)} = \frac{1}{C'} a_{\vartheta\mu} \frac{dy^\mu}{dt} \frac{dy^\lambda}{dt} t^\vartheta_{(\alpha)} + p^\lambda \frac{C_{(\alpha)}}{K'}$$

$$\text{or} \quad C_{(\alpha)} p^\lambda - K' \left[t^\lambda_{(\alpha)} - \frac{1}{C'} a_{\vartheta\mu} \frac{dy^\mu}{dt} \frac{dy^\lambda}{dt} t^\vartheta_{(\alpha)} \right] = 0$$

as the differential equation of the union curves in the subspace K_{2m} of M_{2n+1}.

Let us now assume that M_{2n+1} admits of a Riemannian metric g_{ij}. The differential equation of the union curve in the hypersurface K_{2n} of M_{2n+1} will now take the form

$$q^a - k_n \left[\frac{s^a}{r} - \frac{s^d}{r} g_{bd} \frac{dy^b}{ds} \frac{dy^a}{ds} \right] = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

wherein k_n is the normal component of the curvature vector and s is the arc length. If we denote the left-hand side of eqn. (19) by λ^a, then λ^a will give the components of the union curvature vector and the magnitude of this vector is the union curvature. Thus we find that if the curve C is an asymptotic curve in which case $k_n = 0$ along the curve, then for the union curve ($\lambda^a = 0$), $q^a = 0$ and the curve is a geodesic. Thus if the union curve is a geodesic, then it is either an asymptotic curve or the vector of components

$$v^a = \frac{s^a}{r} - \frac{s^d}{r} g_{bd} \frac{dy^b}{ds} \frac{dy^a}{ds}$$

is a null vector.

The union curves in subspace K_{2m} when M_{2n+1} admits of a Riemannian metric, we find on multiplying eqn. (17) by $a_{\lambda\mu}p^{\mu}$ ($a_{\lambda\mu}$ being the induced metric tensor of K_{2m}) that

$$k^{*}_{\alpha} = v_{(\alpha)}k^{*2}_{g}$$

where k^{*}_{α} is the resolved part of curvature vector in the direction of the curve of congruence, k^{*}_{g} is the magnitude of the curvature vector of C in K_{2m}.

Also from eqn. (16) we have

$$v_{(\alpha)} = C_{(\alpha)}/H_{\lambda\mu}\frac{dy^{\lambda}}{ds}\frac{dy^{\mu}}{ds}.$$

Hence

$$k^{*}_{\alpha}/k^{*2}_{g} = C_{(\alpha)}/H_{\lambda\mu}\frac{dy^{\lambda}}{ds}\frac{dy^{\mu}}{ds}$$

or $$C_{(\alpha)}k^{*2}_{g} = H_{\lambda\mu}\frac{dy^{\lambda}}{ds}\frac{dy^{\mu}}{ds}k^{*}_{\alpha}.$$

Hence the necessary and sufficient condition that the union curves of the subspace be the geodesic of the enveloping space is that either of the following holds: (i) $H_{\lambda\mu}\frac{dy^{\lambda}}{ds}\frac{dy^{\mu}}{ds} = 0$ or (ii) the direction of the curves of the congruence are tangential to C. (We have assumed here that $C_{(\alpha)} \neq 0$).

From the well-known result $k^{*2}_{g} = k^{*2}_{n} + k^{2}_{g}$, where k^{*}_{g} and k^{*}_{n} are the geodesic curvature and the normal curvature in K_{2m} and k_{g} in M_{2n+1} and in view of the above theorem we have:

If either of the following two holds, the third also holds:

(i) $H_{\lambda\mu}\frac{dy^{\lambda}}{ds}\frac{dy^{\mu}}{ds} = 0.$

(ii) The direction of the congruence is tangential to C.

(iii) The curve C is both geodesic and asymptotic line in the subspace.

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