HYDROMAGNETIC FLOW NEAR AN ACCELERATED FLAT PLATE

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Unsteady flow of a viscous incompressible and electrically conducting fluid past an infinite flat plate in the presence of a uniform transverse magnetic field is studied for the two cases when the plate is (i) a perfect conductor, (ii) an insulator. The plate is assumed to start at time $t = 0$ with velocity $U(t)$ in its plane. Using the Laplace transform technique general formulae for the velocity, induced magnetic field and current density are obtained as a function of U when the magnetic Prandtl number is unity. The case of uniformly accelerated plate is studied in detail.

1. INTRODUCTION

The flow of a viscous incompressible and electrically conducting fluid past an infinite flat plate in the presence of a uniform transverse magnetic field when the plate executes simple harmonic motion parallel to itself has been discussed by Ong and Nicholls (1959). They have neglected the induced magnetic field. The same problem is studied by Kakutani (1958) when the induced magnetic field is included.

Recently, Gupta (1960) discussed the flow of an electrically conducting viscous incompressible fluid due to the accelerated motion of an infinite flat plate in the presence of a uniform transverse magnetic field. Assuming that the magnetic Reynolds number is small he has neglected the induced magnetic field. He has also assumed that the electric field $\vec{E} = 0$. In view of these simplified assumptions Maxwell's equations become redundant and the problem becomes simple. Ludford (1959) and Chang and Yen (1959) have considered the impulsive motion of a perfectly conducting infinite plate through a conducting fluid in the presence of a transverse applied uniform magnetic field. The plate was taken to move with constant velocity in its own plane and the fluid was assumed to be viscous and incompressible.

In the present paper the flow of a viscous incompressible and electrically conducting fluid past an infinite flat plate in the presence of a uniform transverse magnetic field is considered for the two cases when the plate is (i) perfectly conducting, (ii) non-magnetic and non-conducting. The plate is supposed to start at time $t = 0$ with velocity $U(t)$ along the x-axis in its plane.

General formulae for velocity, induced magnetic field and current density are obtained as a function of U by using the method of Laplace transformation when the magnetic Prandtl number is unity. The case of a uniformly accelerated plate is studied in detail. Graphs are drawn showing the variations of velocity, current density, skin friction and magnetic field.

Similar problems have also been studied by Drake (1961), Soundalgekar (1965), Pop (1968) and some others.

2. Analysis

Taking the x-axis along the plate and y-axis perpendicular to it, the basic equations governing the unsteady flow of a viscous incompressible and electrically conducting fluid past an infinite flat plate in the presence of a uniform transverse magnetic field H_0 are given by (Kakutani 1958; Ludford 1959)

$$\left(\eta \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right)H + H_0 \frac{\partial u}{\partial y} = 0 \quad \ldots \quad (1)$$

$$\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{\mu_e}{2\rho}H^2\right) = 0 \quad \ldots \quad (2)$$

$$A_0^2 \frac{\partial H}{\partial y} + H_0\left(\nu \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right)u = 0 \quad \ldots \quad (3)$$

where $A_0 = \left(\frac{\mu_e}{\rho}\right)^{\frac{1}{2}} H_0$ is the Alfvén velocity and $\eta = \frac{1}{\sigma \mu_e}$ is the magnetic diffusivity.

From Maxwell's equations the components of current density are given by

$$J_x = 0, \quad J_y = 0, \quad J_z = -\left(\frac{\partial H}{\partial y}\right). \quad \ldots \quad (4)$$

The boundary conditions for velocity are

$$u(y, 0) = 0 \quad \text{for } y \geqslant 0 \quad \ldots \quad (5)$$

$$u(0, t) = U(t), \quad u(\infty, t) = 0. \quad \ldots \quad (6)$$

3. Solution of the Problem

Section I: Perfectly Conducting Plate

Following Ludford (1959), the boundary conditions for induced magnetic field are

$$t < 0 : y > 0 : H = 0 \quad \ldots \quad (7)$$

$$t > 0 : y = 0 : \frac{\partial H}{\partial y} = 0 \quad \ldots \quad (8)$$

$$y = \infty : H = \frac{\partial H}{\partial y} = 0 \quad \ldots \quad (9)$$

On taking the Laplace transform of (1), (3) and (4) we get

$$(\eta d^2/dy^2 - p)\bar{H} + H_0\, d\bar{u}/dy = 0 \qquad \ldots \quad (10)$$

$$H_0(\nu d^2/dy^2 - p)\bar{u} + A_0^2\, d\bar{H}/dy = 0 \qquad \ldots \quad (11)$$

and

$$\bar{J}_z = -(d\bar{H}/dy), \qquad \ldots \quad (12)$$

where

$$\bar{u} = \int_0^\infty u\, e^{-pt}\, dt, \quad \bar{H} = \int_0^\infty H\, e^{-pt}\, dt, \quad \bar{J}_z = \int_0^\infty J_z\, e^{-pt}\, dt.$$

Now the boundary conditions (5) to (9) become

$$\left.\begin{aligned} y = 0 : \bar{u} = \bar{U}(t), \quad d\bar{H}/dy = 0 \\ y = \infty : \bar{u} = \bar{H} = \frac{d\bar{H}}{dy} = 0. \end{aligned}\right\} \qquad \ldots \quad (13)$$

Solution of (10) and (11) satisfying the boundary conditions (13) is given by

$$\bar{u} = \frac{\bar{U}(t)}{p(m^2-n^2)}\,[n^2(\eta m^2 - p)\, e^{-my} - m^2(\eta n^2 - p)\, e^{-ny}] \qquad \ldots \quad (14)$$

$$\bar{H} = \frac{H_0\bar{U}(t)}{p} \cdot \frac{mn}{(m^2-n^2)}\,(n\, e^{-my} - m\, e^{-ny}) \qquad \ldots \quad (15)$$

where

$$m = (a+bp)^{\frac{1}{2}} + (a+cp)^{\frac{1}{2}}, \quad n = (a+bp)^{\frac{1}{2}} - (a+cp)^{\frac{1}{2}}$$

with

$$a = A_0^2/4\eta\nu, \quad b = (\eta^{\frac{1}{2}} + \nu^{\frac{1}{2}})^2/4\eta\nu \quad \text{and} \quad c = (\eta^{\frac{1}{2}} - \nu^{\frac{1}{2}})^2/4\eta\nu.$$

Substituting the value of $\bar{H}$ from (15) in (12), we get

$$\bar{J}_z = \frac{\bar{H}_0\bar{U}(t)}{p} \cdot \frac{m^2n^2}{(n^2-m^2)}\,(e^{-my} - e^{-ny}). \qquad \ldots \quad (16)$$

Again from Ohm's law, we get

$$E_x = 0, \quad E_y = 0, \quad E_z = \frac{1}{\sigma}(J_z - \sigma\mu_e\bar{H}_0 u). \qquad \ldots \quad (17)$$

Special case $\eta = \nu$—It can be seen that none of the transforms $\bar{H}$, $\bar{u}$ or $\bar{J}_z$ can, in general, be inverted in terms of known functions. However, when $\eta = \nu$, i.e. $c = 0$, this can be done. Using convolution theorem we get

$$u(y,t) = \frac{1}{4\sqrt{\pi\nu}} \int_0^t U(t-\tau)\left[y\tau^{-\frac{3}{2}}\left\{e^{-\frac{(y+A_0\tau)^2}{4\nu\tau}} + e^{-\frac{(y-A_0\tau)^2}{4\nu\tau}}\right\}\right.$$

$$\left. - A_0\tau^{-\frac{1}{2}}\left\{e^{-\frac{(y+A_0\tau)^2}{4\nu\tau}} - e^{-\frac{(y-A_0\tau)^2}{4\nu\tau}}\right\}\right] d\tau \qquad \ldots \quad (18)$$

$$H(y,t) = \frac{H_0}{4A_0\sqrt{\pi\nu}} \int_0^t U(t-\tau)\left[y\tau^{-\frac{3}{2}}\left\{e^{-\frac{(y+A_0\tau)^2}{4\nu\tau}} - e^{-\frac{(y-A_0\tau)^2}{4\nu\tau}}\right\}\right.$$

$$\left. - A_0\tau^{-\frac{1}{2}}\left\{e^{-\frac{(y+A_0\tau)^2}{4\nu\tau}} + e^{-\frac{(y-A_0\tau)^2}{4\nu\tau}}\right\}\right] d\tau \qquad \ldots \quad (19)$$

$$J_z = \frac{H_0}{4A_0\sqrt{\pi\nu}} \int_0^t U(t-\tau)\Big[\tau^{-\frac{3}{2}}\Big(\frac{y^2}{4\nu}-\frac{\tau}{2}\Big)\Big\{e^{-\frac{(y+A_0\tau)^2}{4\nu\tau}} - e^{-\frac{(y-A_0\tau)^2}{4\nu\tau}}\Big\}$$
$$-\frac{A_0^2}{4\nu}\tau^{-\frac{1}{2}}\Big\{e^{-\frac{(y+A_0\tau)^2}{4\nu\tau}} - e^{-\frac{(y-A_0\tau)^2}{4\nu\tau}}\Big\}\Big]\, d\tau. \quad \ldots \quad (20)$$

If in these results we put $U(t) = U_0$ (constant), then on integration we get the same results as those obtained by Chang and Yen (1959) for their case.

The above results have a compact form but are inconvenient for studying the behaviour near the plate $y \sim 0$ in the general case.

Let us consider the case of a uniformly accelerated plate, i.e. the case when $U(t) = U_0 t$. We then get after integration

$$\Phi(\eta, M) = \frac{4u(y,t)}{U_0 t} = M^2\Big[\Big(1+\eta M-\frac{M^2}{2}\Big)(e^{-2M\eta}-1)\operatorname{erfc}\Big(\eta-\frac{M}{2}\Big)$$
$$+\Big(1+\eta M+\frac{M^2}{2}\Big)(e^{2M\eta}-1)\operatorname{erfc}\Big(\eta+\frac{M}{2}\Big)$$
$$-\frac{2M}{\sqrt{\pi}}\Big\{e^{-(\eta-\frac{M}{2})^2}-e^{-(\eta+\frac{M}{2})^2}\Big\}\Big]$$
$$+\Big(1-\frac{2\eta}{M}\Big)(1+e^{2M\eta})\operatorname{erfc}\Big(\eta-\frac{M}{2}\Big)$$
$$+\Big(1+\frac{2\eta}{M}\Big)(1+e^{2M\eta})\operatorname{erfc}\Big(\eta+\frac{M}{2}\Big) \quad \ldots \quad (21)$$

$$\Phi^*(\eta, M) = \frac{2HA_0}{H_0U_0t} = \frac{1}{M^2}\Big\{(1+e^{-2M\eta})\operatorname{erfc}\Big(\eta-\frac{M}{2}\Big)-(1+e^{2M\eta})\operatorname{erfc}\Big(\eta+\frac{M}{2}\Big)\Big\}$$
$$-\frac{2}{M\sqrt{\pi}}\Big\{e^{-(\eta+\frac{M}{2})^2}+e^{-(\eta-\frac{M}{2})^2}\Big\}$$
$$+\Big(1+\frac{2\eta}{M}\Big)\operatorname{erfc}\Big(\eta+\frac{M}{2}\Big)-\Big(1-\frac{2\eta}{M}\Big)\operatorname{erfc}\Big(\eta-\frac{M}{2}\Big) \quad \ldots \quad (22)$$

$$J = \frac{2A_0^2}{H_0U_0}J_z = \frac{1}{M\sqrt{\pi}}\Big\{(1+e^{-2M\eta})e^{-(\eta-\frac{M}{2})^2}-(1+e^{2M\eta})e^{-(\eta+\frac{M}{2})^2}\Big\}$$
$$-(1-e^{-2M\eta})\operatorname{erfc}\Big(\eta-\frac{M}{2}\Big)-(1-e^{2M\eta})\operatorname{erfc}\Big(\eta+\frac{M}{2}\Big) \quad \ldots \quad (23)$$

where $\eta = y/2\sqrt{\nu t}$ and $M = A_0\sqrt{t/\nu}$.

The skin friction at the plate is given by

$$\tau_\omega = \rho\nu\Big(\frac{\partial u}{\partial y}\Big)_{y=0} + \mu_e H_0 (H)_{y=0}$$

or

$$\tau_\omega^* = \frac{\tau_\omega}{\rho A_0 U_0 t} = -\operatorname{erfc}\frac{M}{2} - \frac{2}{\sqrt{\pi}}\frac{e^{-M^2/4}}{M}. \qquad (24)$$

Section II: Non-conducting Plate

For a non-conducting plate equations of motion (1) to (3) remain unchanged and following Kakutani (1958) the boundary conditions for induced magnetic field can be written as

$$\left.\begin{aligned} t \leqslant 0 : y \geqslant 0 : H = 0 \\ t > 0 : y = 0 : H = 0 \\ y = \infty : H = 0 \end{aligned}\right\}. \qquad (25)$$

Proceeding as in section I the solution of eqns. (1), (3) and (4) satisfying the boundary conditions (5), (6) and (25) in the case when $\eta = \nu$ is given by

$$u(y,t) = \frac{y \cosh(A_0 y/2\nu)}{2\sqrt{\pi\nu}} \int_0^t U(t-\tau)\tau^{-\frac{3}{2}} \exp\left(-\frac{A_0^2}{4\nu}\tau - \frac{y^2}{4\nu\tau}\right) d\tau \qquad (26)$$

$$H(y,t) = -\frac{H_0 y \sinh(A_0 y/2\nu)}{2\sqrt{\pi\nu}} \int_0^t U(t-\tau)\tau^{-\frac{3}{2}} \exp\left(-\frac{A_0^2}{4\nu}\tau - \frac{y^2}{4\nu\tau}\right) d\tau \qquad (27)$$

$$J_z = \frac{H_0}{A_0\sqrt{\pi\nu}} \int_0^t U(t-\tau)\left[\frac{A_0 y \cosh(A_0 y/2\nu)}{4\nu}\tau^{-\frac{3}{2}} \exp\left(-\frac{A_0^2}{4\nu}\tau - \frac{y^2}{4\nu\tau}\right)\right.$$
$$\left. - \sinh(A_0 y/2\nu)\tau^{-\frac{5}{2}}\left(\frac{y^2}{4\nu} - \frac{\tau}{2}\right) \exp\left(-\frac{A_0^2}{4\nu}\tau - \frac{y^2}{4\nu\tau}\right)\right] d\tau. \qquad (28)$$

For uniformly accelerated plate, i.e. when $U(t) = U_0 t$, we have

$$\Phi(\eta, M) = \frac{4u(y,t)}{U_0 t} = \left(1 - \frac{2\eta}{M}\right)(1 + e^{-2M\eta}) \operatorname{erfc}\left(\eta - \frac{M}{2}\right)$$
$$+ \left(1 + \frac{2\eta}{M}\right)(1 + e^{2M\eta}) \operatorname{erfc}\left(\eta + \frac{M}{2}\right) \qquad (29)$$

$$\Phi^*(\eta, M) = \frac{2HA_0}{H_0 U_0 t} = \frac{1}{2}\left[\left(1 - \frac{2\eta}{M}\right)(e^{-2M\eta} - 1) \operatorname{erfc}\left(\eta - \frac{M}{2}\right)\right.$$
$$\left. - \left(1 + \frac{2\eta}{M}\right)(e^{2M\eta} - 1) \operatorname{erfc}\left(\eta + \frac{M}{2}\right)\right] \qquad (30)$$

$$J = \frac{2A_0^2}{H_0 U_0} J_z = \frac{M^4}{4}\left\{\left(1 - \frac{2\eta}{M}\right)(1 + e^{-2M\eta}) \operatorname{erfc}\left(\eta - \frac{M}{2}\right)\right.$$
$$\left. + \left(1 + \frac{2\eta}{M}\right)(1 + e^{2M\eta}) \operatorname{erfc}\left(\eta + \frac{M}{2}\right)\right\}$$

$$+\frac{1}{2}\left\{\left(1-\eta M+\frac{M^2}{2}\right)(e^{-2M\eta}-1)\operatorname{erfc}\left(\eta-\frac{M}{2}\right)\right.$$
$$\left.+\left(1+\eta M+\frac{M^2}{2}\right)(e^{2M\eta}-1)\operatorname{erfc}\left(\eta+\frac{M}{2}\right)\right\}$$
$$+\frac{M}{\sqrt{\pi}}\left\{e^{-\left(\eta+\frac{M}{2}\right)^2}-e^{-\left(\eta-\frac{M}{2}\right)^2}\right\}. \qquad (31)$$

The skin friction at the plate is given by

$$\tau^*_\omega = -\frac{1}{2}\left(1+\frac{2}{M^2}\right)\operatorname{erfc}\frac{M}{2}-\frac{1}{\sqrt{\pi}}\frac{e^{-M^2/4}}{M} \qquad (32)$$

4. Discussion

Figures 1, 2 and 3 show respectively the variation of dimensionless velocity, induced magnetic field and current density with η for different values of M in the case of uniformly accelerated plate.

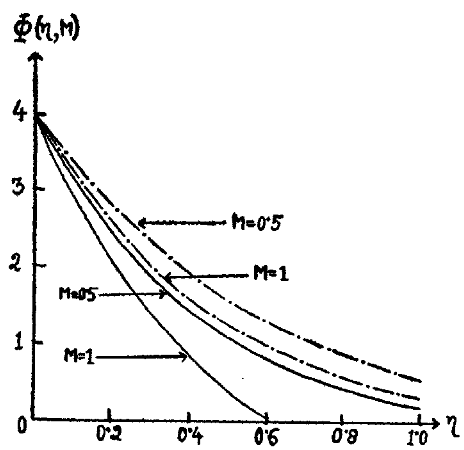

Fig. 1. Velocity profile versus η (—— perfectly conducting plate, –·–·– non-conducting plate)

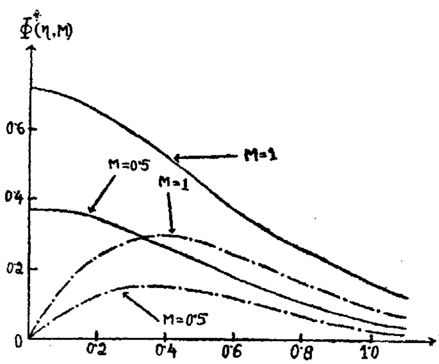

Fig. 2. Induced magnetic field profile versus η (—— perfectly conducting plate, –·–·– non-conducting plate)

It is seen that the velocity decreases with M while the induced magnetic field and current density increase as M increases. Further it is also seen that the velocity is greater for a non-conducting plate than for a plate which is a perfect conductor; whereas the induced magnetic field is less for a non-conducting plate than for a perfectly conducting plate. The current density does not exhibit any such marked behaviour.

The following table shows the variation of τ^*_ω for different values of M.

A study of the table indicates that the skin friction coefficient decreases as the magnetic field strength M is increased. Further it is also seen that the skin friction is greater in the case of a perfectly conducting plate than in the case of a non-conducting plate.

M	τ^*_ω for perfectly conducting plate	τ^*_ω for non-conducting plate
0·2	−5·6974	−5·6582
0·4	−2·9329	−2·8580
0·6	−2·0466	−1·9354
0·8	−1·6301	−1·4842
1·0	−1·3991	−1·2199

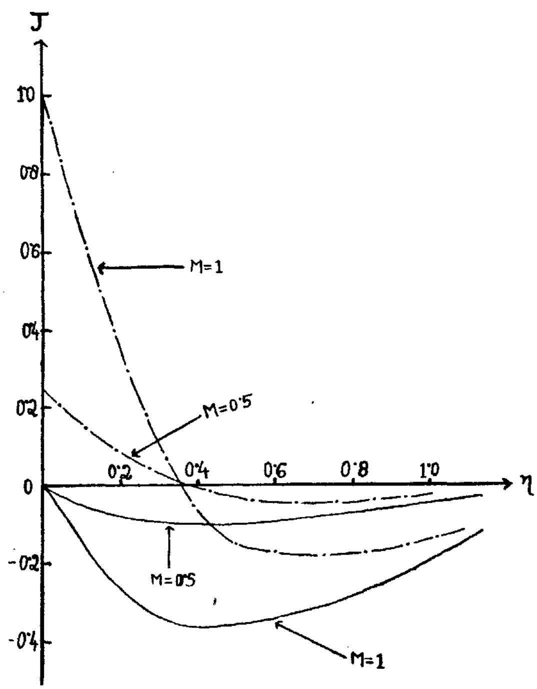

FIG. 3. Current density profile versus η (——perfectly conducting plate, –·–·– non-conducting plate)

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