PRESSURE SHOCKS IN DISSIPATIVE MAGNETOGASDYNAMICS

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The propagation of the pressure shocks through a finitely electrically conducting, viscous and heat-conducting gas in the presence of a uniform magnetic field has been studied. The novel features of the problem are the anisotropic nature of the wave propagation and the existence of a simple relation between the jumps of vorticity and the current density.

INTRODUCTION

Thomas and Edstram (1961) have pointed out the possibility of the existence of pressure shocks in viscous heat-conducting gases. These are characterized by the continuity of velocity across the shock while a discontinuity in pressure is admitted. Also the pressure gradient along the normal to the shock surface in the medium immediately behind the shock is taken to be zero while the temperature is taken to be stationary. The significance of such shocks in the consideration of blast waves is also noted there.

In the present paper we shall treat the problem of propagation of pressure shocks into an ideal gas which is uniform and at rest in the presence of a uniform magnetic field. The gas is assumed to be viscous, heat-conducting and has a finite electrical conductivity. We assume the existence of a singular surface $\Sigma(t)$ across which the velocity is continuous, while the fluid pressure and magnetic field (tangential) are discontinuous so that the 'total pressure', the sum of the fluid pressure and magnetic pressure, is discontinuous. Also we assume that the gradient of the 'total pressure' along the normal to the shock surface is zero immediately behind the shock. These imply

$$u_i' = 0; \quad \left(\frac{dp^*}{dn}\right)' = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

in which we have put $p^* = p+\mu_e H^2/2$, where p denotes the fluid pressure, μ_e the magnetic permeability, H_i and U_i $(i = 1, 2, 3)$ the components of the magnetic field and velocity respectively. The derivative of p^* with respect to n indicates its derivative along the unit positive normal n_i to the shock in the direction of its propagation. The dash denotes the value of the quantities in question, immediately behind the shock.

Using the compatibility and shock relations we obtain the jumps of the field variables. The equation of state in conjunction with assumption (1) now leads to an expression for the velocity of propagation of the shock. We hope to consider the growth of this shock in a later communication.

In contrast to the purely gasdynamical case (Thomas and Edstram 1961), we note here that the singular surface is rotational and there is a simple relation between the jumps of current density and vorticity which is analogous to that obtained by Kanwal and Truesdell (1960).

Basic Equations

The equations governing the coupled motion of magnetogasdynamics under the aforementioned assumptions are (Pai 1957)

$$\frac{\partial \rho}{\partial t}+\rho_{,i}u_i+\rho u_{i,i}=0 \quad \ldots \quad (2)$$

$$H_{i,i}=0 \quad \ldots \quad (3)$$

$$\frac{\partial H_i}{\partial t}-u_{i,j}H_j+H_{i,j}u_j+H_i u_{k,k}-\frac{1}{\sigma\mu_e}H_{i,jj}=0 \quad \ldots \quad (4)$$

$$\rho\frac{\partial u_i}{\partial t}+\rho u_k u_{i,k}-\tau_{ij,j}+\mu_e H_k H_{k,i}-\mu_e H_{i,j}H_j=0 \quad \ldots \quad (5)$$

$$\frac{\partial(\rho U)}{\partial t}+S_{i,i}-S^*_{j,j}=0 \quad \ldots \quad (6)$$

$$p=J\rho(c_p-c_v)T \quad \ldots \quad (7)$$

where we have put

$$\tau_{ij}=\mu(u_{i,j}+u_{j,i})-(p+\tfrac{2}{3}\mu u_{k,k})\delta_{ij} \quad \ldots \quad (8)$$

$$U=\tfrac{1}{2}u^2+Jc_vT+\tfrac{1}{2}\mu_e H^2/\rho \quad \ldots \quad (9)$$

$$S_i=\rho u_i(\tfrac{1}{2}u^2+Jc_vT)-\tau_{ij}u_j+\mu_e H^2 u_i \quad \ldots \quad (10)$$

$$S^*_i=\mu_e H_i H_k u_k+JKT_{,i}+\frac{1}{\sigma}\left(\tfrac{1}{2}H^2_{,i}+H_jH_{i,j}\right). \quad \ldots \quad (11)$$

The quantities occurring in the above equations are the components of magnetic field H_i, velocity components u_i, the fluid pressure p, the density ρ and the temperature T. The magnetic permeability μ_e, the electrical conductivity σ, the coefficient of viscosity μ, the mechanical equivalent of heat J and the specific heats at constant volume c_v and at constant pressure c_p are all assumed to be constants. The usual summation convention is employed and the comma denotes the differentiation with respect to the coordinates x_i ($i = 1, 2, 3$) of a rectangular system, as assumed dynamically admissible, to which the motion of the gas is referred. Here x_i, being rectangular coordinates, no distinction is being made between contravariant and covariant components of the field variables.

Shock and Compatibility Conditions

It can be easily shown (Thomas 1949; Kanwal 1960) that the shock conditions corresponding to eqns. (2)–(6) are

$$[\rho v_n] = 0 \quad \ldots \quad (12)$$

$$[H_n] = 0 \quad \ldots \quad (13)$$

$$[H_i v_n] = [H_n u_i] + \frac{1}{\sigma \mu_e} [H_{i,j}] n_j \quad \ldots \quad (14)$$

$$\rho' v_n [v_i] = [\tau_{ij}] n_j - \tfrac{1}{2}\mu_e [H^2] n_i + \mu_e H_{on} [H_i] \quad \ldots \quad (15)$$

$$\rho' v_n [\tfrac{1}{2}u^2 + J c_v \dot{T} + \mu_e H^2/2\rho] + \frac{\mu_e}{2} [H^2 u_i] n_i - [\tau_{ij} u_j] n_j - \mu_e [H_i u_i H_j] n_j$$

$$-KJ[T_{,i}] n_i - \frac{1}{\sigma} \left([\tfrac{1}{2} H^2_{,j}] n_j - [H_j H_{i,j}] n_j \right) = 0 \quad \ldots \quad (16)$$

where we have put $v_i = u_i - G n_i$; G denotes the velocity of the shock surface $\Sigma(t)$ in the direction of its unit normal vector n_i. In eqns. (14)–(16) use of the relations (12) and (13) has been made. In the above equations the bracket stands for the value of the quantity in question immediately behind the shock surface $\Sigma(t)$ minus its value in front of it.

It can be easily seen that if the gas is devoid of the dissipative effects then the shock relations 12–16 reduce to those obtained by Lust (1953) and Kanwal (1960). Also in the presence of the dissipative effects and in the absence of magnetic field these reduce to those given by Thomas and Edstram (1961).

Since $[u_i] n_i = 0$ and $G \neq 0$, by hypothesis, it follows from (12) that $[\rho] = 0$. Due to the continuity of velocity and density across $\Sigma(t)$, the compatibility conditions of the first order for these quantities have the following form (Thomas 1949, 1957)

$$[u_{i,j}] = \lambda_i n_j; \quad \left[\frac{\partial u_i}{\partial t}\right] = -G\lambda_i \quad \ldots \quad (17)$$

$$[\rho_{,i}] = \xi n_i; \quad \left[\frac{\partial \rho}{\partial t}\right] = -G\xi \quad \ldots \quad (18)$$

where λ_i and ξ are functions defined over the surface $\Sigma(t)$. Now combining eqns. (15) with eqn. (8) and making use of the first set of eqns. (17), we get

$$[p^*] n_i = \mu \lambda_i + \tfrac{1}{3}\mu \lambda_k n_k n_i + \mu_e H'_n [H_i] \quad \ldots \quad (19)$$

in which use has been made of the relation $[v_i] = 0$, which follows from the hypothesis.

If we represent, parametrically, the moving surface by functions $x_i\,(y^1, y^2, t)$, $(i = 1, 2, 3)$ and denote $x_{i\alpha}$ the derivatives of the space coordinates x_i with respect to the surface coordinates y^α; $\alpha = 1, 2$ then the latter quantities for α fixed are the components of a vector in the space which is tangent to the surface $\Sigma(t)$. Clearly y^{α} cannot be, in general, orthogonal and so we distinguish between covariant and contravariant components. Also $a_{\alpha\beta} = x_{i\alpha}x_{i\beta}$ is the surface matrix.

The normal components of eqn. (19) give

$$[p^*] = \tfrac{4}{3}\mu\lambda_n. \qquad (20)$$

In view of the fact that the velocity vanishes on the surface $\Sigma(t)$ and from eqns. (13)–(16) we get the following relations:

$$[H_{i,j}]n_j = -\sigma\mu_e G[H_i] \qquad (21)$$

$$\rho_0 GJc_v[T]+\frac{\mu_e}{2}G[H^2]+KJ[T,_i]n_i+\frac{1}{\sigma}\left(\tfrac{1}{2}[H^2_{,j}]n_j-[H_jH_{i,j}]n_i\right)=0. \qquad (22)$$

Now let all vector quantities be decomposed, with respect to the shock surface, into normal component and tangential components along the coordinates on the shock surface. Thus any vector quantity q_i can be written as

$$q_i = q_n n_i + q^{\alpha}x_{i\alpha} \qquad (23)$$

where

$$q_{\beta} = q_i x_{i\beta} \qquad (24)$$

and

$$q^{\beta} = a^{\alpha\beta}q_{\alpha}. \qquad (25)$$

From eqn. (19) we get†

$$\mu_e H_{0n}[H_i] = -\mu\lambda^{\alpha}x_{i\alpha}. \qquad (26)$$

Thus we obtain from (20) and (26) an expression for the jump of the pressure as

$$[p] = \tfrac{4}{3}\mu\left(\lambda_n+\frac{3}{4}\,\frac{\lambda^{\alpha}H^1_{\alpha}}{H_{0n}}-\frac{3}{8}\,\frac{\mu\lambda_{\alpha}\lambda^{\alpha}}{\mu_e(H_{0n})^2}\right). \qquad (27)$$

Also from eqns. (7), (27) and the relation (26), we obtain

$$[T] = \frac{4}{3}\,\frac{\mu}{J\rho_0(c_p-c_v)}\left(\lambda_n+\frac{3}{4}\,\frac{H^1_{\alpha}\lambda^{\alpha}}{H_{0n}}-\frac{3}{8}\,\frac{\mu\lambda_{\alpha}\lambda^{\alpha}}{\mu_e H^2_{0n}}\right). \qquad (28)$$

PLANE SHOCKS

The expression for the velocity of propagation G of the pressure shocks can be derived exactly in the same way as adopted by Thomas and Edstram (1961). However, the algebra involved in its calculations is formidable, so we now confine attention only to plane shocks. Thus we assume that the velocity and the magnetic field vectors have only two components each along the x- and y-axis.

Taking the jumps of each term in eqn. (2) and using the compatibility conditions (17) and (18) and also taking into consideration that $[\rho] = 0$ and $[u_i] = 0$, we get

$$\xi G = \lambda_1\rho_0. \qquad (29)$$

† When $\theta = 90°$ the shock front turns out to be an irrotational one.

Differentiating eqn. (7) and then taking jumps, we have

$$[p,_1] = J\rho_0(c_p-c_v)([T,_1]+[T\rho,_1]). \quad \ldots \quad (30)$$

By making use of the well-known identity

$$[PQ] = [P]\,[Q]+Q_0[P]+P_0[Q] \quad \ldots \quad (31)$$

and the condition (1), eqn. (30) can be written as

$$J(c_p-c_v)\left([T,_1]+\frac{T_0}{\rho_0}[\rho,_1]+\frac{1}{\rho_0}[T]\,[\rho,_1]\right)+\frac{\mu_e}{2}[H^2_{,1}] = 0. \quad \ldots \quad (32)$$

From eqns. (21), (22), (29) and (32), we get an expression for G as ‡

$$G^2 = \frac{K\lambda_1\left(p_0+\frac{4}{3}\mu\lambda_1-\frac{\mu_e}{2}[H_2^2]\right)}{\rho_0(c_p-c_v)\left\{\mu_e(H_{02}\eta+\eta^2)\left(\frac{K}{\rho_0(c_p-c_v)\nu_H}-1\right)+\frac{\left(\frac{4}{3}\mu\lambda_1+\frac{(c_p-2c_v)}{c_v}\mu_e(H_{02}\eta+\frac{1}{2}\eta^2)\right)}{(c_p-c_v)/c_v}\right\}} \quad \ldots \quad (33)$$

where $\nu_H = \dfrac{1}{\sigma\mu_e}$ and $\eta = [H_2]$.

From eqns. (17), (21) and (26) a simple but important relation between the jumps of vorticity W_3 and the current density J_3 is obtained as

$$H_{01}[J_3] = \sigma\mu G[W_3]. \quad \ldots \quad (34)$$

It is clear from the expressions (33) and (34) that the wave propagation is anisotropic and the shock wave is a rotational one. To understand the nature of wave propagation eqn. (33) is transformed to the non-dimensional form

$$M^2 = \frac{\gamma\cos\phi\left(\frac{4}{3}\chi\cos\phi+\frac{1}{\gamma}+\chi\sin\phi\tan\theta-\frac{\chi^2\sin^2\phi}{2\beta^2\cos^2\theta}\right)}{P\left\{\frac{4}{3}\cos\phi+\frac{\chi\sin\phi}{\beta^2\cos^2\theta}\left(\frac{\gamma R}{P}-\frac{\gamma}{2}\right)-\sin\phi\tan\theta\left(\frac{\gamma R}{P}-1\right)\right\}} \quad \ldots \quad (35)$$

wherein the use of eqn. (26) is made and also the following parameters are introduced:

$$\left.\begin{aligned} &\lambda_1 = \lambda\cos\phi,\ \lambda_2 = \lambda\sin\phi,\ \mu\lambda = \gamma\chi p_0,\ \gamma = c_p/c_v,\ M^2 = \frac{G^2}{a_0^2},\ a_0^2 = \frac{\gamma p_0}{\rho_0}, \\ &P = \frac{\mu c_p}{K},\ R = \frac{\mu}{\rho_0\nu_H},\ \beta^2 = \frac{\mu H_0^2}{\gamma p_0},\ H_{01} = H_0\cos\theta,\ H_{02} = H_0\sin\theta. \end{aligned}\right\} \quad (35a)$$

In eqn. (34) if we take $\eta > 0$ and $H_{01} > 0$, i.e. $\phi < 0$ $(-\frac{\pi}{2} < \phi < 0)$, then we call the discontinuity surface as 'fast pressure shock' (M_f). On the other hand, if we take $\eta < 0$ and $H_{01} > 0$, i.e. $\phi > 0$ $(0 < \phi < \pi/2)$, then we name the

‡ The symmetry of G with θ is evident from eqn. (33).

discontinuity as 'slow pressure shock' (M_s). The variation of the speeds of the shocks with respect to θ is shown in Fig. 1.§ Further it is observed that there can be two types of slow pressure shocks.

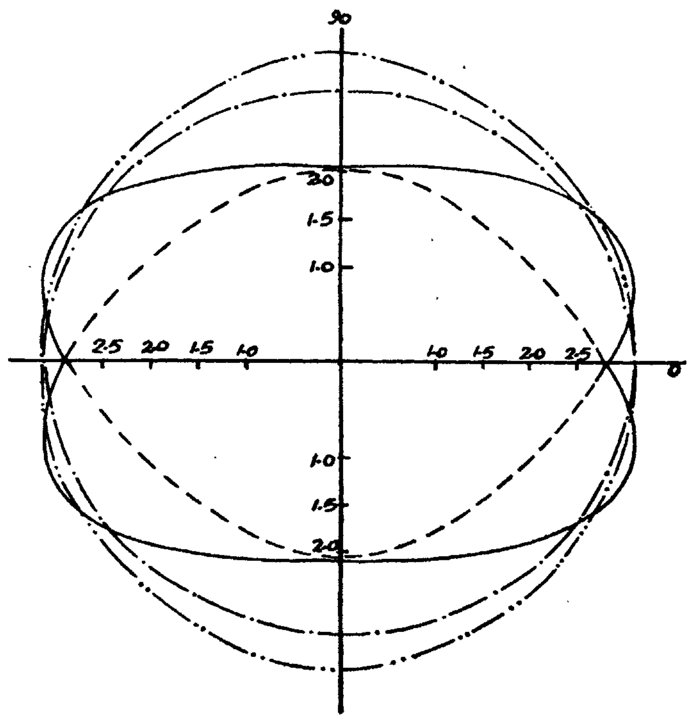

FIG. 1. Variation of the shock velocity with the angle which the direction of the applied magnetic field makes with the positive x-axis for $\beta^2 = 0{\cdot}5$, $\chi = 2{\cdot}0$, $P = 0{\cdot}692$ and $R = 17{\cdot}6/70$ [—Fast ($\phi = -60$); --- slow ($\phi = 60$); -·- slow Alfvén ($\phi = 60$); -··- slow switch-off ($\phi = 60$)].

(i) Type 1 M_s shocks when $H_2' = H_{02}$

(ii) Type 2 M_s shocks when $H_2' = 0$.

The types 1 and 2 M_s shocks may be respectively called as Alfvén and switch-off pressure shocks. From eqn. (33) the speeds of these shocks are given, respectively, as

$$M^2 = \frac{\gamma\chi \cos\phi(1/\gamma + 4/3\chi\cos\phi)}{P(\gamma-1)\left(\frac{4}{3}\chi\frac{\cos\phi}{(\gamma-1)} + 2\beta^2\sin^2\theta\left(\frac{\gamma R}{P(\gamma-1)} - 1\right)\right)} \quad \cdots (36)$$

§ The numerical work was carried out on CDC 3,600 computer at Tata Institute of Fundamental Research.

$$M^2 = \frac{\gamma\chi \cos\phi\left(\frac{1}{\gamma}+\frac{4}{3}\chi\cos\phi+\frac{\beta^2}{2}\sin^2\theta\right)}{P(\gamma-1)\left(\frac{4}{3}\chi\frac{\cos\phi}{(\gamma-1)}-\frac{(\gamma-2)}{(\gamma-1)}\frac{\beta^2}{2}\sin^2\theta\right)}. \quad \ldots \quad \ldots \quad (37)$$

The variations of these speeds with respect to θ are shown in Fig. 1.

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