INFINITESIMAL GEOMETRY IN HILBERT SPACE—III

by N. N. Ghosh, *Indian Association for the Cultivation of Science, Calcutta 32*

(Communicated by S. N. Bose, F.N.I.)

(*Received 1 May 1969*)

This paper deals with the general properties of curves lying either in an m-dimensional linear subspace within an infinitesimal domain of a real Hilbert space, or lying on an r-dimensional curved manifold immersed in the linear subspace. The linear subspace may be chosen in a variety of ways discussed in Part II. For simplicity of treatment the linear subspace considered here is generated by a set of m elementary infinitesimal first order vectors defined in Part I. The analytical process followed is an extension of that adopted by the author (Ghosh 1938) in developing the geometry of curves in hyperspace.

§ 1. Let

$$\rho = \sum_{i=1}^{m} s_i \alpha^i \quad \ldots \quad (1.1)$$

represent an m-dimensional linear subspace generated by a system of m orthonormal infinitesimal vectors α^i passing through the origin in a real Hilbert space characterized by an infinity of mutually orthogonal coordinate axes. We have then

$$\begin{Bmatrix} \alpha^k \\ \alpha^l \end{Bmatrix} = \int_a^b f^k(x) f^l(x)\, dx = 1, \quad \text{if } k = l \\ \qquad = 0, \quad \text{if } k \neq l \Bigg\} \quad k, l = 1, 2, \ldots m. \quad \ldots \quad (1.2)$$

To represent a curve lying in the linear subspace we take the scalar variables s_i to be functions of a single parameter t so that the parametric equation of the curve becomes

$$s_i = \sigma_i(t), \quad i = 1, 2, \ldots m.$$

Choosing a definite value t of the parameter at a point P_0 let us form the vector

$$A(t) = \sum_{i=1}^{m} \sigma_i(t) \alpha^i \quad \ldots \quad (1.3)$$

the tangent vector at P_0 is then

$$A' = \frac{d}{dt} A(t) \quad \ldots \quad (1.4)$$

the linear element of the curve at P_0 being given by

$$ds^2 = \begin{Bmatrix} A' \\ A' \end{Bmatrix} dt^2. \quad \ldots \quad (1.5)$$

The osculating subspace of kth order at P_0 is formed by the vectors $A', A'', \ldots \partial^k A$, where

$$\partial^k A = \frac{d^k}{dt^k} A(t). \quad \ldots \quad (1.6)$$

We shall now determine a set of normals to the curve at the point P_0.

Starting with the tangent vector A' and taking A'' we form the first normal vector

$$\nu_1 = A'' - A' \begin{Bmatrix} A'' \\ A' \end{Bmatrix} \Big/ \begin{Bmatrix} A' \\ A' \end{Bmatrix} \quad \ldots \quad (1.7)$$

which is perpendicular to A' and lies in the osculating plane at P_0.
Next, the vector

$$\nu_2 = A''' - A'' \begin{Bmatrix} A'A''' \\ A'A'' \end{Bmatrix} \Big/ \Delta^{(2)} - A' \begin{Bmatrix} A'''A'' \\ A'A'' \end{Bmatrix} \Big/ \Delta^{(2)} \quad \ldots \quad (1.8)$$

where

$$\Delta^{(2)} = \begin{Bmatrix} A'A'' \\ A'A'' \end{Bmatrix}$$

is the second normal, which is perpendicular to the osculating plane and lies in the osculating 3-space at P_0.

Generally,

$$\nu_k = \partial^{k+1} A - \sum_{j=1}^{k} \partial^j A \begin{pmatrix} \partial^j A \\ \partial^{k+1} A \end{pmatrix} \Delta^{(k)} / \Delta^{(k)} \quad \ldots \quad (1.9)$$

is the kth normal perpendicular to the osculating k-space. In the above $\Delta^{(k)}$ denotes the scalar determinant

$$\begin{Bmatrix} A'A'' \ldots \partial^k A \\ A'A'' \ldots \partial^k A \end{Bmatrix} \quad \ldots \quad (1.10)$$

and $\begin{pmatrix} \partial^j A \\ \partial^{k+1} A \end{pmatrix} \Delta^{(k)}$ denotes the scalar determinant $\Delta^{(k)}$ in which $\partial^j A$ in the first row is replaced by $\partial^{k+1} A$. It may be noted here that the scalar product of $\partial^k A$ and $\partial^l A$ is given by

$$\begin{Bmatrix} \partial^k A \\ \partial^l A \end{Bmatrix} = \sum_{i=1}^{m} \frac{d^k}{dt^k} \sigma_i \frac{d^l}{dt^l} \sigma_i. \quad \ldots \quad (1.11)$$

One can see from (1.9) that

$$\begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix} = \begin{Bmatrix} \nu_k \\ \partial^{k+1} A \end{Bmatrix} = \Delta^{(k+1)} / \Delta^{(k)}. \quad \ldots \quad (1.12)$$

If $\delta\theta_k$ denote the angle between consecutive osculating k-spaces at P_0 then it can be proved (Ghosh 1938)

$$\left(\frac{d\theta_k}{dt}\right)^2 = \begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix} \Big/ \begin{Bmatrix} \nu_{k-1} \\ \nu_{k-1} \end{Bmatrix}. \quad \ldots \quad (1.13)$$

The set of m vectors $\nu_0 = A'$, $\nu_1\nu_2 \ldots \nu_{m-1}$ mutually orthogonal will form a new frame of reference associated with the particular point P_0 on the curve. The vector $\partial^k A$, when $k \leq m$, is then expressible in the form

$$q_{k0}\nu_0+q_{k1}\nu_1+\cdots+q_{k,k-1}\nu_{k-1} \qquad \ldots \quad (1.14)$$

and, when $k > m$, in the form

$$q_{k0}\nu_0+q_{k1}\nu_1+\cdots+q_{k,m-1}\nu_{m-1}. \qquad \ldots \quad (1.15)$$

The coefficient q_{kl} is given by

$$\begin{Bmatrix}\partial^k A \\ \nu_l\end{Bmatrix} = q_{kl}\begin{Bmatrix}\nu_l \\ \nu_l\end{Bmatrix}. \qquad \ldots \quad (1.16)$$

Let the linear subspace at P_0 generated by the set of m mutually orthogonal vectors ν_k imbedding the curve be represented by

$$P = A(t)+\sum_{h=0}^{m-1} p_h\nu_h. \qquad \ldots \quad (1.17)$$

Then in the neighbourhood of P_0 the new parametric equation of the curve is (Ghosh 1938)

$$p_h = \frac{u^{h+1}}{(h+1)!}q_{h+1,h}+\frac{u^{h+2}}{(h+2)!}q_{h+2,h}+\cdots \quad (h = 0, 1, 2, \ldots m-1) \qquad (1.18)$$

where u is the new parameter.

§ 2. Let us now consider the angle $\delta\phi_k$ between two consecutive kth normals. We have

$$\cos^2\delta\phi_k = \begin{Bmatrix}\nu_k(t+\delta t) \\ \nu_k\end{Bmatrix}^2 \Bigg/ \begin{Bmatrix}\nu_k \\ \nu_k\end{Bmatrix}\begin{Bmatrix}\nu_k(t+\delta t) \\ \nu_k(t+\delta t)\end{Bmatrix} \qquad \ldots \quad (2.1)$$

whence, retaining terms up to δt^2,

$$\left(\frac{d\phi_k}{dt}\right)^2 = \begin{Bmatrix}\nu_k\,\nu_k' \\ \nu_k\,\nu_k'\end{Bmatrix} \Bigg/ \begin{Bmatrix}\nu_k \\ \nu_k\end{Bmatrix}^2. \qquad \ldots \quad (2.2)$$

Now differentiating (1.9) we get

$$\nu_k' = \delta^{k+2}A-\sum_{j=1}^{k}\delta^{j+1}A\begin{pmatrix}\delta^j A \\ \delta^{k+1}A\end{pmatrix}\Delta^{(k)}/\Delta^{(k)}-\sum_{j=1}^{k}\delta^j A\frac{d}{dt}\left[\begin{pmatrix}\delta^j A \\ \delta^{k+1}A\end{pmatrix}\Delta^{(k)}/\Delta^{(k)}\right]. \qquad \ldots \quad (2.3)$$

But referred to the orthogonal frame at P_0 we can write

$$\nu_k' = p_{k0}\nu_0+p_{k1}\nu_1+\cdots+p_{k,m-1}\nu_{m-1} \qquad \ldots \quad (2.4)$$

whence

$$\begin{Bmatrix}\nu_k' \\ \nu_l\end{Bmatrix} = p_{kl}\begin{Bmatrix}\nu_l \\ \nu_l\end{Bmatrix}, \quad \begin{Bmatrix}\nu_k' \\ \nu_k\end{Bmatrix} = p_{kk}\begin{Bmatrix}\nu_k \\ \nu_k\end{Bmatrix}.$$

From (2.3) it follows that

$$\begin{Bmatrix}\nu_k' \\ \nu_l\end{Bmatrix} = 0, \quad l \geq k+2.$$

Moreover, $\begin{Bmatrix} \nu_k \\ \nu_l \end{Bmatrix}$ being zero ($k \neq l$) we get

$$\begin{Bmatrix} \nu_k' \\ \nu_l \end{Bmatrix} = -\begin{Bmatrix} \nu_l' \\ \nu_k \end{Bmatrix}.$$

Considering the above one can write (2.4) in the reduced form

$$\nu_k' = p_{k,\,k-1}\nu_{k-1} + p_{k,\,k}\nu_k + p_{k,\,k+1}\nu_{k+1} \qquad \dots \qquad \dots \qquad (2.5)$$

where

$$p_{k,\,k+1} = 1$$

$$p_{k,\,k-1} = -\begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix} \Big/ \begin{Bmatrix} \nu_{k-1} \\ \nu_{k-1} \end{Bmatrix} = -\left(\frac{d\theta_k}{dt}\right)^2$$

$$\begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix} p_{k,\,k} = \begin{Bmatrix} \partial^{k+2}A \\ \nu_k \end{Bmatrix} - \begin{Bmatrix} \partial^{k+1}A \\ \nu_k \end{Bmatrix} \begin{Bmatrix} \partial^{k+1}A \\ \nu_{k-1} \end{Bmatrix} \Big/ \begin{Bmatrix} \nu_{k-1} \\ \nu_{k-1} \end{Bmatrix}$$

or

$$p_{k,\,k} = q_{k+2,\,k} - q_{k+1,\,k-1}.$$

Using (2.5) in (2.2) we get

$$\left(\frac{d\phi_k}{dt}\right)^2 = \left(\frac{d\theta_k}{dt}\right)^2 + \left(\frac{d\theta_{k+1}}{dt}\right)^2. \qquad \dots \qquad \dots \qquad \dots \qquad (2.6)$$

Let λ denote a unit vector independent of t, then cosine of the angle between ν_k and λ is

$$l_k = \begin{Bmatrix} \nu_k \\ \lambda \end{Bmatrix} \Big/ \begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix}^{\frac{1}{2}}. \qquad \dots \qquad \dots \qquad \dots \qquad (2.7)$$

Differentiating the above we have

$$\frac{d}{dt} l_k = \begin{Bmatrix} \nu_k' \\ \lambda \end{Bmatrix} \begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix}^{-\frac{1}{2}} - \begin{Bmatrix} \nu_k \\ \lambda \end{Bmatrix} \begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix}^{-\frac{3}{2}} \begin{Bmatrix} \nu_k' \\ \nu_k \end{Bmatrix}.$$

Making use of (2.5) the above gives

$$\frac{d}{dt} l_k = -\frac{d\theta_k}{dt} l_{k-1} + \frac{d\theta_{k+1}}{dt} l_{k+1}. \qquad \dots \qquad \dots \qquad \dots \qquad (2.8)$$

It may be noted that if ν_k is replaced by $\begin{Bmatrix} \nu_k \\ \nu_k \end{Bmatrix}^{\frac{1}{2}} \bar{\nu}_k$, $\bar{\nu}_k$ being the unit vector along ν_k, (2.5) yields the relation

$$\bar{\nu}_k' = -\frac{d\theta_k}{dt} \bar{\nu}_{k-1} + \frac{d\theta_{k+1}}{dt} \bar{\nu}_{k+1}. \qquad \dots \qquad \dots \qquad \dots \qquad (2.9)$$

§ 3. To define curves on an r-dimensional manifold immersed in the linear subspace

$$\rho = \sum_{i=1}^{m} S_i \alpha^i$$

satisfying (1.2) we take the parametric equation of the manifold to be

$$S_i = \sigma_i(u_1, u_2, \ldots u_r) i = 1, 2, \ldots r. \quad \ldots \quad \ldots \quad (3.1)$$

The functions σ_i of the u's being analytic in a given region, a point Q_0 on the curve will then be given by the vector

$$\sum_{i=1}^{m} \sigma_i(u_1, u_2, \ldots u_r)\, \alpha^i \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

where the u's are functions of a single variable t.
Denoting the vector (3.2) by $B(t)$, the tangent vector of the curve at Q_0 is

$$B'(t) = \sum_{i=1}^{m} \sum_{h=1}^{r} \frac{\partial \sigma_i}{\partial u_h} \frac{du_h}{dt} \alpha^i \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

where the system of r vectors

$$C_h = \sum_{i=1}^{m} \frac{\partial \sigma_i}{\partial u_h} \alpha^i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

forms the tangent subspace of the manifold at Q_0.

Adopting the usual summation convention with regard to repeated indices let us write (3.3) as

$$B'(t) = C_{h_1} u'_{h_1}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

By successive differentiations we get

$$B''(t) = C_{h_1} u''_{h_1} + C_{h_1 h_2} u'_{h_1} u'_{h_2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

$$B'''(t) = C_{h_1} u'''_{h_1} + 3C_{h_1 h_2} u''_{h_1} u'_{h_2} + C_{h_1 h_2 h_3} u'_{h_1} u'_{h_2} u'_{h_3} \quad \ldots \quad \ldots \quad (3.7)$$

and so on. In the above C's are symmetric in respect of the subscripts.

We now replace the systems of mutually independent vectors

$$C_{h_1}, C_{h_1 h_2} (h_1 \leqslant h_2), \ldots C_{h_1 h_2 \ldots h_k} (h_1 \leqslant h_2 \leqslant \ldots \leqslant h_k)$$

associated with the point Q_0 of the manifold by starred ones according to the scheme laid down in one of my earlier papers (Ghosh 1940). As basis for the osculating subspace of kth order at Q_0 we shall take the systems of vectors $C_{h_1}, C^*_{h_1 h_2} \ldots C^*_{h_1 h_2 \ldots h_k}$ defining the tangent subspace and successive normal subspaces of the manifold at Q_0. Let the corresponding conjugate system be denoted by $C_{(h_1)}, C^*_{(h_1 h_2)} \ldots C^*_{(h_1 h_2 \ldots h_k)}$. Then the vector $\partial^k B$ is expressible in the form (m being sufficiently large)

$$\partial^k B = C_{h_1} \begin{Bmatrix} C_{(h_1)} \\ \partial^k B \end{Bmatrix} + C^*_{h_1 h_2} \begin{Bmatrix} C^*_{(h_1 h_2)} \\ \partial^k B \end{Bmatrix} + \ldots + C^*_{h_1 h_2 \ldots h_k} \begin{Bmatrix} C^*_{(h_1 h_2 \ldots h_k)} \\ \partial^k B \end{Bmatrix}. \quad (3.8)$$

The restriction $h_1 \leqslant h_2 \leqslant h_3 \ldots \leqslant h_k$ on the subscripts of C^*'s imposes an obvious modification in the usual summation convention.

Applying (1.9)

$$\nu_k = \partial^{k+1}B - \sum_{j=1}^{k} \partial^j B \begin{pmatrix} \partial^j B \\ \partial^{k+1}B \end{pmatrix} \Delta_B^{(k)} / \Delta_B^{(k)}$$

where

$$\Delta_B^{(k)} = \begin{Bmatrix} B'B'' \ldots \partial^k B \\ B'B'' \ldots \partial^k B \end{Bmatrix}$$

$\nu_k \cdot \Delta_B^{(k)}$ can, therefore, be expressed as

$$C_{h_1} \begin{pmatrix} C_{(h_1)} \\ \partial^{k+1}B \end{pmatrix} \Delta_B^{(k+1)} + C^*_{h_1h_2} \begin{pmatrix} C^*_{(h_1h_2)} \\ \partial^{k+1}B \end{pmatrix} \Delta_B^{(k+1)} + \ldots C^*_{h_1h_2\ldots h_k} \begin{pmatrix} C^*_{(h_1h_2\ldots h_k)} \\ \partial^{k+1}B \end{pmatrix} \Delta_B^{(k+1)}. \quad \ldots \quad (3.9)$$

To obtain the equation of the geodesic at the point Q_0 on the r-dimensional manifold we change the parameter t into s given by the equation

$$s = \int \begin{Bmatrix} B' \\ B' \end{Bmatrix}^{\frac{1}{2}} dt. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

Denoting the successive fundamental vectors of the geodesic by $\beta', \beta'' \ldots \beta^{(k)}$ we have then

$$\begin{Bmatrix} \beta' \\ \beta' \end{Bmatrix} = 1, \quad \begin{Bmatrix} \beta'' \\ \beta' \end{Bmatrix} = 0 \quad \text{and so on.}$$

For a geodesic

$$\begin{Bmatrix} C_{(h)} \\ \beta'' \end{Bmatrix} = 0, \quad h = 1, 2, \ldots r. \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

Now

$$\beta' = C_{k_1} \frac{du_{k_1}}{ds}$$

$$\beta'' = C_{k_1} \frac{d^2u_{k_1}}{ds^2} + C_{k_1k_2} \frac{du_{k_1}}{ds} \cdot \frac{du_{k_2}}{ds}$$

and so on.

Hence (3.11) can be written in the standard form

$$\frac{d}{ds} \begin{Bmatrix} C_{(k)} \\ \beta' \end{Bmatrix} + \begin{Bmatrix} C_{k_1k_2} \\ C_{(k)} \end{Bmatrix} \begin{Bmatrix} C_{(k_1)} \\ \beta' \end{Bmatrix} \begin{Bmatrix} C_{(k_2)} \\ \beta' \end{Bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (3.12)$$

where

$$\begin{Bmatrix} C_{(k)} \\ \beta' \end{Bmatrix} = \frac{du_k}{ds}, \quad k = 1, 2, \ldots r.$$

ACKNOWLEDGEMENT

The author wishes to thank Professor S. N. Bose, F.N.I., for his kind interest and helpful comments.

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