SOME CRITERIA OF STEWART-TOWNSEND AND OGURA-MEAKODA EQUATIONS FOR DECAY OF ISOTROPIC TURBULENCE

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The self-preserving solutions for Heisenberg's equation for decay of isotropic turbulence have been obtained by Heisenberg (1948) and generalized by Sen (1951). This self-preserving solution given by Sen also satisfies the more general form of decay equation due to Karman (cf. Ghosh 1954). In this present note, the self-preserving solutions of the decay equations due to (i) Stewart and Townsend (1951), and (ii) Ogura and Meakoda (1953) have been obtained. These solutions are the same as those due to Sen (1951). The asymptotic nature of these solutions for small and large wavenumbers as given by the respective equation has also been discussed. For $k \to 0$, we get Loitsiansky's result, while for $k \to \infty$, we obtain Kolmogoroff's spectrum.

INTRODUCTION

The decay of turbulence according to Heisenberg (1948) is given by

$$-\frac{\partial}{\partial t}\int_0^k F(k', t)\, dk' = 2(\nu+\eta_k)\int_0^k F(k', t)k'^2\, dk' \quad \ldots \quad \ldots \quad (1)$$

where $F(k, t)$ is the spectrum function of turbulence, ν the kinematic viscosity and η_k the turbulent viscosity. Heisenberg assumed η_k to be of the form

$$K\int_k^\infty \sqrt{\frac{F(k', t)}{k'^3}}\, dk'.$$

For this η_k, the above decay equation admits of a self-preserving solution of the form

$$F(k, t) = \frac{1}{\sqrt{t}} f(k, \sqrt{t}).$$

But at the initial stage, when the low frequency eddies are predominant (ν small compared to η_k), a more general type of self-preserving solution of (1) was given by Sen (1951) as

$$F(k, t) = \frac{1}{K^2 k_0^3 t_0^2}\left(\frac{t_0}{t}\right)^{2-3c} f\left(\left(\frac{k}{k_0}\right)\left(\frac{t}{t_0}\right)^c\right)$$

where c is an arbitrary constant and restriction imposed on c being $c < \frac{2}{3}$. It was shown by Sen that this solution has the characteristics

$$f(x) = \text{const}\; x^{\frac{2-3c}{c}} \quad \text{as } x \to 0$$

and

$$f(x) = \text{const}\; x^{-\frac{5}{3}} \quad \text{as } x \to \infty$$

whatever be the value of c (with implicit restriction $c < \frac{2}{3}$).

The present note deals with the self-preserving solutions of equations for decay of isotropic turbulence, one of the equations being due to Stewart and Townsend (1951) and the other due to Ogura and Meakoda (1953). Both the equations admit of self-preserving solutions of the form given by Sen (1951). Further, the asymptotic behaviours of these solutions for wave-numbers $k \to 0$ and $k \to \infty$ are found to be the same as obtained for equations due to Heisenberg (1948) and Karman (1948).

Case I—Stewart and Townsend Equation

Stewart and Townsend equation (1951) for the decay of isotropic turbulence is given by

$$-\frac{\partial}{\partial t}\int_0^k F(k', t)\, dk' = 2(\nu+\gamma_{\mathrm{ST}}) \left\{ \int_k^\infty \frac{[F(k', t)]^{\frac{1}{2l}}}{k'^{1+\frac{1}{2l}}} \right\}^l \int_0^k F(k', t) k'^2\, dk'. \qquad (2)$$

At the initial stage, when ν is small compared to γ_{ST}, let us assume a self-preserving solution of the form

$$F(k, t) = \frac{1}{\gamma_{\mathrm{ST}}^2 k_0^3 t_0^2}\left(\frac{t_0}{t}\right)^{2-3c} f\left(\left(\frac{k}{k_0}\right)\left(\frac{t}{t_0}\right)^c\right). \qquad \ldots \quad \ldots \quad (3)$$

Substituting eqn. (3) in eqn. (2) (when viscosity term is neglected), we get

$$2(1-c)\int_0^x f(x)\, dx - cxf(x) = \left\{ \int_x^\infty \frac{f^{\frac{1}{2l}}(x)}{x^{1+\frac{1}{2l}}}\, dx \right\}^l \int_0^x x^2 f(x)\, dx. \qquad \ldots \quad (4)$$

It can be established that eqn. (3) admits of similar solutions of the form $h^3 f(hx)$ when $f(x)$ is any solution of the same equation and h is any constant unconnected with l. We know that this type of similarity property holds good for the solutions of Heisenberg's equation.

Let us now examine the behaviour of $f(x)$, satisfying eqn. (4), when $x \to 0$ and also when $x \to \infty$.

(i) *Asymptotic Behaviour of $f(x)$ at $x \to 0$*

Let

$$f(x) \sim Ax^n, \quad A \neq 0 \quad (x \to 0). \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

Then from (4), we get the following equation for n:

$$n^2 c(1-l) + n\{2c(1-2l) + 2(l-1)\} + (2-3c)(1+l) = 0. \qquad \ldots \quad (6)$$

If n_1 and n_2 be the roots of ~~this equation~~, then

$$n_1 = \frac{2-3c}{c} \qquad \ldots \quad (6a)$$

and

$$n_2 = \frac{1+l}{1-l}. \qquad \ldots \quad (6b)$$

Sen (1951) and Ghosh (1954) obtained the asymptotic solution (6*a*) for Heisenberg and Karman spectrum function respectively. Putting $c = \frac{1}{2}$, we get $n = 1$, corroborating the result due to Heisenberg. Further, putting $c = \frac{2}{7}$, we get $n = 4$ and arrive at Loitsiansky's result.

For the second solution, we see that in order that n_2 may be positive, $l < 1$ and putting $l = \frac{3}{5}$, one can get back Loitsiansky's (1938) result.

(ii) *Asymptotic Behaviour of $f(x)$ at $x \to \infty$*

Putting $f(x') = e^{-\omega(x')}$ in eqn. (4) and following Heisenberg's method of approximation, the behaviour of $f(x)$ for $x \to \infty$ will be given by

$$(2-3c)f(x) - cxf'(x) = \frac{(2l)^l f^{l+\frac{1}{2}}(x)\, x^{-\frac{1}{2}}}{\{f(x)-xf'(x)\}^l}\, x^2 f(x)$$

$$+\frac{d}{dx}\left[\frac{(2l)^l f^{l+\frac{1}{2}}(x) x^{-\frac{1}{2}}}{\{f(x)-xf'(x)\}^l}\right]\int_0^x x^2 f(x)\, dx \qquad \ldots \quad (7)$$

(restriction being $\omega'+1 > 0$, dash meaning differentiation with respect to $\log x$).

Let us now assume a solution of the form

$$f(x) \sim Ax^{-n} \quad (x \to \infty)$$

where $A \neq 0$ and n is positive.

From eqn. (7), we have

$$A[(2-3c)+nc]x^{-n} - \frac{A^{\frac{3}{2}}(2l)^l}{(1+n)^l}\left[1+\frac{n+1}{2(n-3)}\right]x^{-\frac{3}{2}(n-1)} = 0. \qquad \ldots \quad (8)$$

If the first term of eqn. (8) be significant, i.e. $n < \frac{3}{2}(n-1)$, viz. $n > 3$, equating the coefficient of x^{-n} to zero, we have for n, the solution $n = \dfrac{3c-2}{c}$.

If the second term be predominant, i.e. $n > \frac{3}{2}(n-1)$, viz. $n < 3$, equating the coefficients of $x^{-\frac{3}{2}(n-1)}$, we have for n,

$$1+\frac{n+1}{2(n-3)} = 0$$

i.e. $n = \frac{5}{3}$.

Since c is restricted to be less than $\frac{2}{3}$, so $n = \dfrac{3c-2}{c}$ is a violation of the condition $n > 3$. Hence we have the only asymptotic solution for x as $f(x) = Ax^{-\frac{5}{3}}$. This is a well-known Kolmogoroff spectrum (1941).

CASE II—OGURA AND MEAKODA EQUATION

Ogura and Meakoda equation (1953) for the decay of isotropic turbulence is given by

$$-\frac{\partial}{\partial t}\int_0^k F(k', t)\, dk' = 2(\nu+\gamma_{\mathrm{OM}})\frac{1}{k}\left[\int_k^\infty F(k')\, dk'\right]^{\frac{1}{2}}\int_0^k F(k', t)k'^2\, dk'. \qquad (9)$$

At the initial stage, when ν is small compared to γ_{OM}, we assume a self-preserving solution of the form

$$F(k, t) = \frac{1}{\gamma_{\mathrm{OM}}^2 k_0^3 t_0^2}\left(\frac{t_0}{t}\right)^{2-3c} f\left(\left(\frac{k}{k_0}\right)\left(\frac{t}{t_0}\right)^c\right). \qquad \ldots \quad \ldots \quad (10)$$

Substituting eqn. (10) in eqn. (9), when viscosity term is neglected, we have

$$2(1-c)\int_0^x f(x)\, dx - cxf(x) = \frac{1}{x}\left[\int_x^\infty f(x)\, dx\right]^{\frac{1}{2}}\int_0^x x^2 f(x)\, dx. \qquad \ldots \quad (11)$$

In this case also, it is seen that eqn. (11) admits of similar solutions of the form $h^3 f(hx)$ where $f(x)$ is any solution of the same equation and h is any arbitrary constant.

Let us now examine the behaviour of $f(x)$, satisfying (11) for two cases: (i) as $x \to 0$ and (ii) as $x \to \infty$.

(i) *Asymptotic Behaviour of $f(x)$ at $x \to 0$*

If $f(x) \sim Ax^n$ as $x \to 0$ $(A \neq 0)$, then from eqn. (11) we get the following equation for n:

$$cn^3 - 2n^2(2c-1) + n(c-2) + 2(2-3c) = 0 \qquad \ldots \quad \ldots \quad (12)$$

or

$$\{cn-(2-3c)\}(n+2)(n-1) = 0.$$

Therefore

$$n_1 = \frac{2-3c}{c}, \quad n_2 = -2, \quad n_3 = 1.$$

Of these three solutions, n_2 being negative is omitted. Here $n_1 = \dfrac{2-3c}{c}$ gives the solution obtained by Sen (1951) and $n_3 = 1$ gives the solution given by Heisenberg. It is also to be noted that $n_1 = 4$ when $c = \frac{2}{7}$. This gives Loitsiansky's (1938) result.

(ii) *Asymptotic Behaviour of $f(x)$ at $x \to \infty$*

Putting $f(x') = e^{-\omega(x')}$ in eqn. (11) and following Heisenberg's method of approximation, the behaviour of $f(x)$ as $x \to \infty$ is given by

$$[(2-3c)+cn]Ax^{-n} = \left[1-\frac{n+1}{2(3-n)}\right]\frac{A^{\frac{3}{2}}}{\sqrt{1-n}}x^{-\frac{3n+3}{2}} \qquad \ldots \quad (13)$$

(restriction being $\omega' > 1$, dash meaning differentiation with respect to $\log x$).

If the first term be significant, then from eqn. (13) we get

$$-n > -\frac{3n+3}{2}, \quad \text{i.e. } n > 3.$$

Accordingly, we have

$$2-3c+cn = 0$$

i.e. $n = \dfrac{3c-2}{c}$.

Again, if we assume the second term to be significant, then $n < 3$ and the vanishing of its coefficient gives $n = \frac{5}{3}$. Thus we obtain Kolmogoroff spectrum.

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