THE CURVATURE OF A CONGRUENCE RELATIVE TO A VECTOR-FIELD OF A FINSLER HYPERSURFACE

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The normal curvature of a surface $x^i = x^i(u^1, u^2)$, $(i = 1, 2, 3)$ of a three-dimensional Euclidean space in the direction of a curve $u^\alpha = u^\alpha(s)$ is given by

$$k_n = \Omega_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}$$

where $\Omega_{\alpha\beta}$ is the second fundamental tensor of the surface. This concept has been generalized by a number of authors such as Kaul (1957), Upadhyay (1964) and Hsiao (1960). Kaul and Hsiao have generalized their results to a hypersurface of a Riemannian space.

In the present paper we have considered a fourfold generalization, to the hypersurface of a Finsler space, of the curvature given by Kaul. This has incidentally led to the generalization of the normal and secondary normal curvatures of Rund (1956), the curvature of a congruence obtained by Prakash (1961) and also the normal curvature of a vector-field given by Prakash (1960).

1. Fundamental Formulae

Let a hypersurface F_{n-1} given by equations $x^i = x^i(u^\alpha)$; $i = 1, \ldots n$; $\alpha = 1, \ldots n-1$ be immersed in an n-dimensional Finsler space F_n. Consider an arbitrary curve $C: u^\alpha = u^\alpha(s)$ of F_{n-1}. The components x'^i $\left(\text{or } \dfrac{dx^i}{ds}\right)$ and u'^α $\left(\text{or } \dfrac{du^\alpha}{ds}\right)$ of the unit tangent vector of C are related by $x'^i = u'^\alpha B^i_\alpha$, where $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$. Let $g_{ij}(x, x')$ be the metric tensor of F_n. The vectors $\underline{n}(u)$ and $\underline{n}^*(u, u')$ consistent with the conditions

$$g_{ij}(x, n)B^i_\alpha n^j = 0, \qquad g_{ij}(x, n)n^i n^j = 1 \quad \ldots \quad (1.1a)$$

and

$$g_{ij}(x, x')B^i_\alpha n^{*j} = 0, \quad g_{ij}(x, n^*)n^{*i} n^{*j} = 1 \quad \ldots \quad (1.1b)$$

are called the normal vector and the secondary normal vector of the hypersurface respectively. The metric tensors $g_{\alpha\beta}(u, u')$ and $\gamma_{\alpha\beta}(u)$ of the hypersurface are such that

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x')B^i_\alpha B^j_\beta \quad \text{and} \quad \gamma_{\alpha\beta}(u) = g_{ij}(x, n)B^i_\alpha B^j_\beta .$$

The induced connection parameter $\Gamma^{*\epsilon}_{\alpha\beta}$ of F_{n-1} is defined by

$$\Gamma^{*\epsilon}_{\alpha\beta}(u, u') = B^{\epsilon}_{i}(B^{i}_{\alpha\beta}+\Gamma^{*i}_{hk}(x, x')B^{h}_{\alpha}B^{k}_{\beta}) \quad \text{(1.2)}$$

where

$$B^{\epsilon}_{i} = g^{\epsilon\alpha}(u, u')g_{ij}(x, x')B^{j}_{\alpha}, \quad B^{i}_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$$

and Γ^{*i}_{hk} are the connection coefficients of the embedding space (Rund 1959, p. 71). The parameter $\Gamma^{*\epsilon}_{\alpha\beta}$ has been used in the definition of what have been called the 'tensor (or mixed) derivative' and 'induced covariant derivative' of the hypersurface. In particular, the tensor derivative of B^{i}_{α} is given by

$$I^{i}_{\alpha\beta} = B^{i}_{\alpha;\beta} = B^{i}_{\alpha\beta}-B^{i}_{\gamma}\Gamma^{*\gamma}_{\alpha\beta}(u, u')+\Gamma^{*i}_{hk}B^{h}_{\alpha}B^{k}_{\beta} \quad \text{(1.3)}$$

whence we have

$$\Omega_{\alpha\beta} = n_i I^{i}_{\alpha\beta}. \quad \text{(1.4)}$$

$\Omega_{\alpha\beta}(u, u')$ are the components of the second fundamental tensor of the hypersurface. Further

$$I^{i}_{\alpha\beta} = \Omega^{*}_{\alpha\beta}n^{*i} \quad \text{(1.5)}$$

where $\Omega^{*}_{\alpha\beta}(u, u')$ are called the components of the secondary second fundamental tensor of the hypersurface.

The tensor derivatives of the vectors n^i and n^{*i} are such that

$$n^{i}_{;\beta} = A^{\alpha}_{\beta}B^{i}_{\alpha}+V_{\beta}n^{i} \quad \text{(1.6a)}$$

$$n^{*i}_{;\beta} = A^{*\alpha}_{\beta}B^{i}_{\alpha}+V^{*}_{\beta}n^{*i} \quad \text{(1.6b)}$$

where

$$A^{\delta}_{\beta} = -\gamma^{\alpha\delta}(\Omega_{\alpha\beta}+E_{ij\beta}B^{i}_{\alpha}n^{j}); \qquad V_{\beta} = n_i n^{i}_{;\beta} \quad \text{(1.7a)}$$

and

$$A^{*\delta}_{\beta} = -g^{\alpha\delta}(\psi\Omega^{*}_{\alpha\beta}+E^{*}_{ij\beta}B^{i}_{\alpha}n^{*j}); \; V^{*}_{\beta} = \frac{1}{\psi}n^{*}_{i}n^{*i}_{;\beta} \quad \text{(1.7b)†}$$

The normal curvature (Rund 1956) of the hypersurface in the direction of C is given by

$$K = n_i \frac{\delta x'^i}{\delta s} \quad \text{(1.8)}$$

where $\delta/\delta s$ is defined by Rund (1959, p. 59).

2. Curvature of a Congruence Relative to a Vector-field

Consider a vector $\underline{\lambda}(u)$ representing a congruence of curves, in F_n, and a vector-field $\underline{v}(u)$ of the hypersurface. At the points of the hypersurface, we

† The tensors $g^{\alpha\beta}$, $\gamma^{\alpha\beta}$ are defined by $g^{\alpha\beta}g_{\alpha\epsilon} = \delta^{\beta}_{\epsilon} = \gamma^{\alpha\beta}\gamma_{\alpha\epsilon}$ and

$$E_{ij\beta} \overset{\text{def}}{=} g_{ij;\beta}(x, n), \quad E^{*}_{ij\beta} = g_{ij;\beta}(x, x').$$

may write

$$\lambda^i = p^\alpha B^i_\alpha + rn^i \quad (2.1a) \quad \text{and} \quad v^i = v^\alpha B^i_\alpha. \qquad (2.1b)$$

These vectors are normalized by the conditions

$$g_{ij}(x, \lambda)\lambda^i\lambda^j = 1 \quad (2.2a) \quad \text{and} \quad g_{ij}(x, v)v^iv^j = g_{\alpha\beta}(u, v)v^\alpha v^\beta = 1. \qquad (2.2b)$$

We define

$$^vK_\lambda = -v^i\frac{\delta\lambda_i}{\delta s}, \quad (2.3a) \qquad ^v\bar{K}_\lambda = -v_i\frac{\delta\lambda^i}{\delta s} \qquad (2.3b)$$

where

$$\lambda_i = g_{ij}(x, \lambda)\lambda^j \quad (2.4a) \quad \text{and} \quad v_i = g_{ij}(x, v)v^j. \qquad (2.4b)$$

The scalar $^vk_\lambda$ will be called the P-curvature (with respect to C) of the congruence $\underline{\lambda}$ relative to the vector-field $\underline{v}$ while $^v\bar{K}_\lambda$ will be called the S-curvature (with respect to C) of the congruence $\underline{\lambda}$ relative to $\underline{v}$.

From (2.1a), (2.4a), (1.6a) and the definition $\bar{E}_{ij;\beta} \overset{\text{def}}{=} g_{ij;\beta}(x, \lambda)$, we get

$$\frac{\delta\lambda_i}{\delta s} = \lambda_{i;\beta}\frac{du^\beta}{ds} = \left(q^\alpha_{ij\beta}B^j_\alpha + p_{ij\beta}n^j + g_{ij}(x, \lambda)p^\alpha I^j_{\alpha\beta}\right)\frac{du^\beta}{ds}, \qquad (2.5)$$

where

$$q^\alpha_{ij\beta} = \left[\bar{E}_{ij\beta}p^\alpha + g_{ij}(x, \lambda)\left(p^\alpha_{;\beta} + rA^\alpha_\beta\right)\right],$$

$$p_{ij\beta} = [r\bar{E}_{ij\beta} + g_{ij}(x, \lambda)(r_{;\beta} + rV_\beta)].$$

Substituting from (2.1b) and (2.5) in (2.3a) and simplifying, we get

$$^vK_\lambda = \hat{\Omega}_{\beta\gamma}(u, u')v^\beta\frac{du^\gamma}{ds}, \qquad (2.6)$$

where

$$\hat{\Omega}_{\beta\gamma}(u, u') = -\left(q^\alpha_{ij\gamma}B^j_\alpha B^i_\beta + p_{ij\gamma}B^i_\beta n^j + g_{ij}(x, \lambda)p^\alpha I^j_{\alpha\gamma}B^i_\beta\right). \qquad (2.7)$$

More generally, if v^i does not satisfy (2.2b), we have

$$^vK_\lambda = \frac{\hat{\Omega}_{\beta\gamma}v^\beta\dfrac{du^\gamma}{ds}}{(g_{\alpha\beta}(u, v)v^\alpha v^\beta)^{\frac{1}{2}}}. \qquad (2.8)$$

We proceed to consider the following particular cases:

(i) Suppose that the congruence $\underline{\lambda}(u)$ is the normal $\underline{n}(u)$ of the hypersurface. This gives $p^\alpha = 0$ and $r = 1$. A simple calculation based on eqns. (1.7a) and (2.7) yields $\hat{\Omega}_{\beta\gamma} = \Omega_{\beta\gamma}$. Hence

$$^vK_\lambda = \Omega_{\beta\gamma}v^\beta\frac{du^\gamma}{ds} \qquad (2.9)$$

which is the normal curvature of the vector-field (Prakash 1960). If, in addition, $v^i = x'^i$ we have $^vK_\lambda = \Omega_{\beta\gamma}\dfrac{du^\beta}{ds}\dfrac{du^\gamma}{ds}$. This is the normal curvature (Rund 1956) of the hypersurface in the direction of the curve.

(ii) Let the vector-field be tangent to the curve, i.e. $v^i = x'^i$.
Equation (2.6) reduces to

$$^vK_\lambda = \hat{\Omega}_{\beta\gamma}(u, u') \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \qquad \ldots \quad \ldots \quad \ldots (2.10)$$

which is the curvature of the congruence (Prakash 1961).

(iii) Suppose that the spaces F_n and F_{n-1} are Riemannian, we have then $\bar{E}_{ij\beta} = 0$, $V_\beta = 0$, $A^\alpha_\beta = -g^{\alpha\delta}\Omega_{\delta\beta}$ (Mishra 1965). Substituting these values and simplifying with the help of (2.6), (2.7) and the relation given immediately after (2.5), we get

$$^vK_\lambda = (r\Omega_{\alpha\beta} - p_{\alpha;\,\beta})v^\alpha \frac{du^\beta}{ds} \qquad \ldots \quad \ldots \quad \ldots (2.11)$$

which is the normal curvature of a vector-field as given by Kaul (1957).†

From (2.1a), (1.6a) and the relation $I^i_{\alpha\beta} = \Omega_{\alpha\beta}n^i - \Omega_{\alpha\beta}n^h B^\delta_h B^i_\delta$ (Rund 1956), we get

$$\frac{\delta\lambda^i}{\delta s} = \lambda^i_{;\beta}\frac{du^\beta}{ds} = \left[(p^\alpha_{;\beta} + rA^\alpha_\beta - p^\epsilon\Omega_{\epsilon\beta}n^h B^\alpha_h)B^i_\alpha + (r_{;\beta} + rv_\beta + p^\epsilon\Omega_{\epsilon\beta})n^i\right]\frac{du^\beta}{ds}. \quad (2.12)$$

We define

$$g_{ij}(x, v)B^i_\alpha B^j_\beta = \bar{\gamma}_{\alpha\beta} \text{ and } g_{ij}(x, v)B^i_\alpha n^j = \bar{\gamma}_\alpha. \qquad \ldots \quad \ldots (2.13)$$

Substituting from (2.4b), (2.12) in (2.3b) and simplifying with the help of (2.13), we have

$$^v\bar{K}_\lambda = \bar{\Omega}_{\alpha\beta}v^\alpha \frac{du^\beta}{ds} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (2.14)$$

where

$$\bar{\Omega}_{\alpha\beta}(u, u') = \left[\bar{\gamma}_{\epsilon\alpha}(\Omega_{\delta\beta}p^\delta n^h B^\epsilon_h - p^\epsilon_{;\,\beta} - rA^\epsilon_\beta) - \bar{\gamma}_\alpha(r_{;\beta} + rv_\beta + p^\delta\Omega_{\delta\beta})\right]. \quad \ldots (2.15)$$

Suppose, in particular, that the spaces F_n and F_{n-1} are Riemannian. We have then

$$\bar{\gamma}_{\alpha\beta} = g_{\alpha\beta}, \;\; \bar{\gamma}_\alpha = 0, \;\; A^\alpha_\beta = -g^{\alpha\delta}\Omega_{\delta\beta}, \;\; v_\beta = 0, \;\; \bar{\gamma}_{\epsilon\alpha}B^\alpha_h n^h = 0 \text{ (Mishra 1965)}$$

and eqn. (2.15) gives $\bar{\Omega}_{\alpha\beta} = (r\Omega_{\alpha\beta} - p_{\alpha;\beta})$. Equation (2.14), thereby, reduces to (2.11). Therefore, the scalar $^v\bar{K}_\lambda$ is another generalization of the curvature given by Kaul.

The curvatures $^vK_\lambda(u, u')$ and $^v\bar{K}_\lambda(u, u')$ studied in this section may be used to define two types of principal directions of a congruence relative to a vector-field.

† The scalar given by (2.11) has been named by Kaul as the generalized normal curvature of a vector-field relative to a congruence. I do not agree with this terminology, since the vector $(\delta\lambda_i/\delta s)$ is a measure of the variation of the congruence $\underline{\lambda}$ and, therefore, the scalar $^vK_\lambda$ given by (2.3a) measures the curvature of the congruence $\underline{\lambda}$ relative to the vector-field $\underline{v}$.

3. Secondary Curvatures

Consider a vector $\underline{\lambda}^*(u, \dot{u})$ defining a congruence of curves in the embedding space and a vector-field $\underline{v}^*(u, \dot{u})$ of the hypersurface. The vector functions $\underline{\lambda}^*(u, \dot{u})$ and $\underline{v}^*(u, \dot{u})$ (single valued in u^α, $\dot{u}^\alpha$) define, in general, two cones of vectors at every point of F_{n-1}. However, in the sequel, only such congruences are considered for which $\dot{u}^\alpha = u'^\alpha$ at the points of C. We may write

$$(a)\ \lambda^{*i} = p^{*\alpha}B^i_\alpha + r^*n^{*i}, \qquad (b)\ v^{*i} = v^{*\alpha}B^i_\alpha. \quad \ldots \quad (3.1)$$

These vectors are normalized by the conditions

$$(a)\ g_{ij}(x, x')\lambda^{*i}\lambda^{*j} = 1, \qquad (b)\ g_{ij}(x, x')v^{*i}v^{*j} = 1. \quad \ldots \quad (3.2)$$

We define

$$(a)\ {}^vK^*_\lambda = -v^{*i}\frac{\delta\lambda^*_i}{\delta s}\Big/\sqrt{\psi}, \quad (b)\ {}^v\bar{K}^*_\lambda = -v^*_i\frac{\delta\lambda^{*i}}{\delta s}\Big/\sqrt{\psi} \quad \ldots \quad (3.3)$$

where

$$\lambda^*_i = g_{ij}(x, x')\lambda^{*j}, \quad v^*_i = g_{ij}(x, x')v^{*j} \text{ and } \psi = g_{ij}(x, x')n^{*i}n^{*j}.$$

The scalar ${}^vK^*_\lambda(u, u')$ will be called P-secondary curvature (with respect to C) of $\underline{\lambda}^*$ relative to $\underline{v}^*$ and ${}^v\bar{K}^*_\lambda(u, u')$ will be called S-secondary curvature (with respect to C) of $\underline{\lambda}^*$ relative to $\underline{v}^*$.

From (3.3a), (3.1a) and (1.6b), we obtain

$${}^vK^*_\lambda = \hat{\Omega}^*_{\alpha\beta}v^\alpha\frac{du^\beta}{ds} \quad \ldots \quad (3.4)$$

where

$$\hat{\Omega}^*_{\alpha\beta}(u, u') = -\psi^{-\frac{1}{2}}\left[E^*_{ij\beta}\lambda^{*j}B^i_\alpha + g_{\alpha\gamma}(u, u')\left(p^{*\gamma}_{;\beta} + r^*A^{*\gamma}_\beta\right)\right]. \quad \ldots \quad (3.5)$$

We shall consider the following particular cases:

(i) Suppose that the congruence $\underline{\lambda}^*$ is along the secondary normal $\underline{n}^*$ of the hypersurface, i.e. $p^{*\alpha} = 0$. The condition (3.2a) implies $r^*\psi^{\frac{1}{2}} = 1$. Therefore, in view of (1.7b), eqn. (3.5) reduces to $\hat{\Omega}^*_{\alpha\beta} = \Omega^*_{\alpha\beta}$ and we have ${}^vK^*_\lambda = \Omega^*_{\alpha\beta}v^{*\alpha}\frac{du^\beta}{ds}$, which is the secondary normal curvature of the vector-field (Prakash 1960). If, in addition, the vector $\underline{v}^*$ is tangent to the curve, we have ${}^vK^*_\lambda = \Omega^*_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}$ which is the secondary normal curvature of the hypersurface (Rund 1956).

(ii) Let the spaces F_n and F_{n-1} be Riemannian. We have then

$$E^*_{ij\beta} = 0,\ \psi = 1,\ V_\beta = 0 \text{ and } A^{*\alpha}_\beta = -\Omega^*_{\delta\beta}g^{\alpha\delta}. \quad \ldots \quad (3.6)$$

Substituting these values in (3.5) we find that eqn. (3.4) reduces to the curvature given by Kaul (refer eqn. (2.11)).

From (3.3*b*) we deduce

$$ {}^{v}\bar{K}^{*}_{\lambda} = \bar{\Omega}^{*}_{\alpha\beta} v^{\alpha} \frac{du^{\beta}}{ds} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.7) $$

where

$$ \bar{\Omega}^{*}_{\alpha\beta} = -\psi^{-\frac{1}{2}} g_{\alpha\gamma}(p^{*\gamma}_{;\,\beta} + r^{*} A^{*\gamma}_{\beta}) . \quad \ldots \quad \ldots \quad \ldots \quad (3.8) $$

It is easy to show that the curvature ${}^{v}\bar{K}^{*}_{\lambda}$ is another generalization, to Finsler hypersurface, of the result of Kaul.

The two secondary curvatures ${}^{v}K^{*}_{\lambda}$ and ${}^{v}\bar{K}^{*}_{\lambda}$ may be used to define two types of asymptotic directions of a congruence relative to a vector-field.

The curvatures obtained in this and the last section reveal the following characteristic features of a Finsler hypersurface.

It is observed in the work of Kaul that the normal curvature of the vector-field relative to a congruence for a surface of a three-dimensional Euclidean space has a unique generalization to the hypersurface of a Riemannian space. The corresponding generalization as obtained in this paper becomes fourfold from Riemannian to Finsler spaces. This fact (the fourfold generalization) may be attributed to following reasons:

(i) The covariant derivative of the metric tensor g_{ij} is not zero.

(ii) There exist two systems of normals $\underline{n}(u)$ and $\underline{n}^{*}(u, u')$ and two second fundamental tensors $\Omega_{\alpha\beta}(u, u')$ and $\Omega^{*}_{\alpha\beta}(u, u')$ for the hypersurface.

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