ON SPECIAL CURVES OF A SUBSPACE OF A FINSLER SPACE

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The K_λ curves of a hypersurface in Riemannian space are studied by Tsagas (1969). The aim of this paper is to study another category of curves which will be called K_λ curves of the Finsler subspace. A curve will be said to be a K_λ curve with respect to a congruence λ (of the enveloping space) if the variety determined by the geodesic curvature vectors of the curve with respect to F_m and F_n contains the vector field λ.

1. Introduction

In the present section we shall outline some basic concepts and fundamental formulae from the theory of Finsler subspace as given by Eliopoulos (1959).

Let an m-dimensional subspace F_m given by equations $x^i = x^i(u^\alpha)$, $(i = 1, \ldots n,\ \alpha = 1, \ldots m)$ be immersed in an n-dimensional Finsler space F_n. Consider a curve $C : x^i = x^i(s)$ of F_m where s denotes the arc length. The components x'^i and u'^α of the unit tangent vector of C are given by $x'^i = X^i_\alpha u'^\alpha$ where $X^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$. The metric tensors $g_{ij}(x, x')$, $g_{\alpha\beta}(u, u')$ of F_n and F_m are such that

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x')X^i_\alpha X^j_\beta. \qquad \ldots \qquad \ldots \qquad \ldots \quad (1.1)$$

The vectors $n^i_{(\nu)}(u)$ $(\nu = m+1, \ldots n)$ and $\overset{*}{n}{}^i_{(\nu)}(u, u')$ $(\nu = m+1, \ldots n)$ satisfying the conditions

$$g_{ij}(x, n_{(\nu)})X^i_\alpha n^j_{(\nu)} = n_{(\nu)i}X^i_\alpha = 0, \quad g_{ij}(x, n_{(\nu)})n^i_{(\nu)}n^j_{(\nu)} = 1 \qquad \ldots \quad (1.2)$$

and

$$g_{ij}(x, x')X^i_\alpha \overset{*}{n}{}^j_{(\nu)} = 0, \quad g_{ij}(x, \overset{*}{n}_{(\nu)})\overset{*}{n}{}^i_{(\nu)}\overset{*}{n}{}^j_{(\nu)} = 1 \qquad \ldots \qquad \ldots \quad (1.3)$$

$g_{ij}(x, x')\overset{*}{n}{}^i_{(\nu)}\overset{*}{n}{}^j_{(\mu)} = \delta^{(\mu)}_{(\nu)}\psi_{(\mu)}$ (no summation with respect to μ) are called, respectively, the normal vectors and the secondary normal vectors of the subspace. The induced connection parameter $\Gamma^{*\epsilon}_{\alpha\beta}(u, u')$ of F_m is given by

$$\Gamma^{*\epsilon}_{\alpha\beta} = X^\epsilon_i\left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^{*i}_{hk}X^h_\alpha X^k_\beta\right),$$

where $X_i^\epsilon = g^{\epsilon\alpha}(u, u')g_{ij}(x, x')X_\alpha^j$ and $\Gamma_{hk}^{*i}(x, x')$ are the connection coefficients of the embedding space F_n.

The parameter $\Gamma_{\alpha\beta}^{*\epsilon}$ is used in the definition of what have been called 'the tensor (or mixed) derivative' and 'the induced covariant derivative' in the subspace. In particular the tensor derivative of X_α^i is given by

$$X_{\alpha\beta}^i = X_{\alpha,\,\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} - X_\gamma^i \Gamma_{\alpha\beta}^{*\gamma} + \Gamma_{hk}^{*i} X_\alpha^h X_\beta^k$$

whence $\Omega_{(\nu)\alpha\beta} = n_{(\nu)i} X_{\alpha\beta}^i$. The $(n-m)$ tensors with the components $\Omega_{(\nu)\alpha\beta}$ are called the second fundamental tensors of the subspace. It is further shown that

$$X_{\alpha\beta}^i = \sum_\nu \overset{*}{\Omega}_{(\nu)\alpha\beta}\, \overset{*i}{n}_{(\nu)}, \qquad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

where $\overset{*}{\Omega}_{(\nu)\alpha\beta}(u, u')$ are the components of the secondary second fundamental tensors of F_m. The following (results given by Eliopoulos 1959) will be used in the sequel. We have

$$X_{\alpha\beta}^i = X_\gamma^i \sum_\nu A_{(\nu)\alpha\beta} M_{(\nu)}^\gamma + \Sigma A_{(\nu)\alpha\beta}\, n_{(\nu)}^i \quad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

where $A_{(\nu)\alpha\beta}$ and $M_{(\nu)}^\alpha$ satisfy

$$\Omega_{(\nu)\alpha\beta} = \sum_\nu A_{(\nu)\alpha\beta} \cos\left(n_{(\mu)}, n_{(\nu)}\right) \text{ and } M_{(\nu)}^\gamma = -n_{(\nu)}^i X_i^\gamma .$$

2. K_λ Curves

Consider a congruence of curves given by vectors $\lambda^i(u)$ such that through each point of F_m there passes one curve of the congruence. The vector field λ can be expressed in the form

$$\lambda^i(u) = t^\alpha X_\alpha^i + \sum_\nu C_{(\nu)} n_{(\nu)} . \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

This is normalized by the condition

$$g_{ij}(x, x')\lambda^i \lambda^j = 1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

Relation (2.2) by virtue of (2.1), (1.1) and (1.2) takes the form

$$g_{\alpha\beta}(u, u')t^\alpha t^\beta = 1 - 2\sum_\nu C_{(\nu)} n_{(\nu)i} \,.\, t^\alpha X_\alpha^i - \sum_{\mu,\nu} C_{(\mu)} C_{(\nu)} g_{ij}(x, x') n_{(\nu)}^i \,.\, n_{(\mu)}^j \quad \ldots \quad (2.3)$$

Let $C : x^i = x^i(s)$ be a K_λ curve on F_m. We have, therefore, $\lambda = wp + uq$, where p, q are the geodesic curvature vectors of C with respect to F_m and F_n respectively and w, u parameters. From the above relation we obtain

$$\lambda^i = \omega p^i + uq^i. \qquad (2.4)$$

Using the well-known relation

$$p^i = p^\alpha X_\alpha^i ; \quad q^i = \left[p^\alpha + \sum_\nu A_{(\nu)\beta\gamma} M_{(\nu)}^\alpha \frac{du^\beta}{ds}\frac{du^\gamma}{ds}\right] X_\alpha^i + \sum_\nu A_{(\nu)\beta\gamma} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} n_{(\nu)}^i \qquad (2.5)$$

where

$$p^\alpha = \frac{d^2u^\alpha}{ds^2} + \gamma^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}$$

equation (2.4) in view of (2.1) reduces to

$$t^\alpha X^i_\alpha + \sum_\nu C_{(\nu)} n^i_{(\nu)} = \left[u \sum_\nu A_{(\nu)\beta\gamma} M^\alpha_{(\nu)} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} + (u+w)p^\alpha\right] X^i_\alpha$$

$$+ u \sum_\nu A_{(\nu)\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} n^i_{(\nu)}. \quad \ldots \quad (2.6)$$

Equating the coefficients of X^i_α and $n^i_{(\nu)}$ respectively we obtain

$$t^\alpha = (w+u)p^\alpha + u \,.\, \sum_\nu A_{(\nu)\beta\gamma} M^\alpha_{(\nu)} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} \text{ and } C_{(\nu)} = u \,.\, A_{(\nu)\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}. \quad (2.7)$$

Multiplying (2.7) by $g_{\alpha\beta}(u, u')p^\beta$ and summing with respect to β we have

$$(u+w) = \left(g_{\beta\delta}t^\beta p^\delta - u \sum_\nu A_{(\nu)\delta\gamma} M^\epsilon_{(\nu)} \frac{du^\delta}{ds}\frac{du^\gamma}{ds} g_{\beta\epsilon}p^\beta\right) k_g^{-2}$$

where k_g is the geodesic curvature of the curve.
Hence (2.7) takes the form

$$k_g^2 t^\alpha - \left(g_{\beta\delta}t^\beta p^\delta - \frac{C_{(\mu)}\sum_\nu A_{(\nu)\delta\gamma}}{A_{(\nu)\rho\sigma}\frac{du^\rho}{ds}\frac{du^\sigma}{ds}}\right) M^\epsilon_{(\nu)} \frac{du^\delta}{ds}\frac{du^\gamma}{ds} g_{\epsilon\beta}p^\beta p^\alpha - k_g^2 C_{(\mu)}\left(A_{(\nu)\rho\sigma}\frac{du^\rho}{ds}\frac{du^\sigma}{ds}\right)^{-1}$$

$$\times \sum A_{(\nu)\beta\gamma} M^\alpha_{(\nu)} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0. \quad \ldots \quad (2.8)$$

From the above we have the following theorem:

Theorem (2.1)—The m-differential equations (2.8) represent K_λ curves relative to the congruence λ.

Particular case: For the hypersurface the congruence λ is given by

$$\lambda^i = t^\alpha X^i_\alpha + C n^i.$$

Equation (2.8) for the K_λ curve takes the form

$$k_g^2 t^\alpha - \left(g_{\beta\delta}t^\beta p^\delta - \frac{C A_{\delta\gamma}\frac{du^\delta}{ds}}{A_{\rho\sigma}\frac{du^\rho}{ds}\frac{du^\sigma}{ds}} \frac{du^\gamma}{ds} g_{\epsilon\beta} M^\epsilon p^\beta\right) p^\alpha - k_g^2 C M^\alpha = 0 \quad \ldots \quad (2.9)$$

which leads to a theorem parallel to Theorem (2.1).

Theorem (2.2)—The differential equations (2.9) represent K_λ curves of the hypersurface.

Let the congruence λ be a normal congruence, i.e. $t^\alpha = 0$ and let $p^\alpha = 0$. Hence we have the following theorem:

Theorem (2.3)—Geodesics of the hypersurface (or subspace) are the only K_λ curves relative to a normal congruence.

3. Secondary K_λ Curves

In this section we shall consider a congruence λ^* which is a function of a line element (u, u'). The vector field λ^* can be represented in the form

$$\lambda^{*i}(u, u') = t^{*\alpha}X^i_\alpha + \Sigma C^*_{(\nu)}(u, u')n^{*i}_{(\nu)} \quad \ldots \quad (3.1)$$

where $C^*_{(\nu)}$ is a scalar and $t^{*\alpha}$ contravariant components of a vector of the subspace. This vector is normalized by the condition

$$g_{ij}(x, x')\lambda^{*i}\lambda^{*j} = 1. \quad \ldots \quad (3.2)$$

Relation (3.2) in view of (3.1) and (1.3) takes the form

$$g_{\alpha\beta}(x, x')t^{*\alpha}t^{*\beta} = 1 - \sum_\mu C^{*2}_{(\mu)}\psi(\mu). \quad \ldots \quad (3.3)$$

The secondary K_λ curves will be defined in the same manner as K_λ curves of the last section. For secondary K_λ curves we have the relation $\lambda^* = w^*p + u^*q$ where w^* and u^* are scalars. From the above relation

$$\lambda^{*i} = w^*p^i + u^*q^i \quad \ldots \quad (3.4)$$

where p^i and q^i are given by

$$p^i = p^\alpha X^i_\alpha, \quad q^i = p^\alpha X^i_\alpha + \sum_{(\nu)} \Omega^*(\nu)\alpha\beta \frac{du^\alpha}{ds}\frac{du^\beta}{ds} n^{*i}_{(\nu)}. \quad \ldots \quad (3.5)$$

Equation (3.4) in view of (3.5) and (3.1) reduces to

$$t^{*\alpha}X^i_\alpha + \sum_\nu C^*_{(\nu)}n^{*i}_{(\nu)} = (u^* + w^*)p^\alpha X^i_\alpha + u^* \sum_\nu \Omega^*_{(\nu)\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} n^{*i}_{(\nu)}. \quad \ldots \quad (3.6)$$

Equating the coefficients of X^i_α and $n^{*i}_{(\nu)}$ respectively, we have

$$t^{*\alpha} = (u^* + w^*)p^\alpha \quad \text{and} \quad C^*_{(\mu)} = u^*\Omega^*_{(\mu)\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}. \quad \ldots \quad (3.7)$$

We shall now consider the following two cases:

Case (i)—Let the congruence λ^* be non-normal. Multiplying (3.7) by $g_{\alpha\beta}t^{*\beta}$ and summing with respect to β, we have after simplification

$$(u^* + w^*) = \Big(1 - \sum_\mu (C^*_{(\mu)})^2\psi(\mu)\Big) \Big/ g_{\alpha\beta}p^\alpha t^{*\beta}.$$

Consequently (3.7) takes the form

$$p^\alpha - t^{*\alpha}g_{\beta\gamma}p^\beta t^{*\gamma}\big(1 - \Sigma C^{*2}_{(\mu)}\psi(\mu)\big)^{-1} = 0. \quad \ldots \quad (3.8)$$

Hence we have the following theorem:

Theorem (3.1)—The differential equations (3.8) determine the secondary K_λ curves on F_m relative to the vector field λ^*.

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For the hypersurface F_{n-1} the congruence λ^* is given by

$$\lambda^{*i} = t^{*\alpha} X^i_\alpha + C n^{*i}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

The equation for the secondary K_λ curve takes the form

$$p^\gamma - g_{\alpha\beta} p^\alpha t^{*\beta} (1 - C^{*2}\psi)^{-1} t^{*\gamma} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

Case (ii)—Let the congruence λ^{*i} be a normal congruence, i.e. $t^{*\alpha} = 0$, so that $p^\alpha = 0$. Hence we have the following theorem:

Theorem (3.2)—Geodesics of the subspace are the only secondary K_λ curves relative to a normal congruence.

4. K_λ Curves and Union Curves

In this section we shall investigate the conditions under which a K_λ curve (or a secondary K_λ curve) is a union curve (Singh 1968).

For a union curve relative to the congruence λ^i we have

$$\lambda^i = A \frac{dx^i}{ds} + B q^i. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

Using (2.1) and (2.5) we obtain

$$\left.\begin{aligned} t^\alpha &= A \frac{du^\alpha}{ds} + B\left(p^\alpha + \sum_\nu M^\alpha_{(\nu)} A_{(\nu)\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds}\right) \\ \text{and} \qquad C_{(\nu)} &= B A_{(\nu)\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \end{aligned}\right] \quad \ldots \quad \ldots \quad (4.2)$$

comparing these with eqns. (2.7), we obtain after some simplification

$$u = B \quad \text{and} \quad w p^\alpha = A \frac{du^\alpha}{ds}.$$

The last relation implies $A = 0$ and $p^\alpha = 0$.

We have, therefore, the following theorem:

Theorem (4.1)—A K_λ curve is a union curve if it is a geodesic of the subspace and in this case the congruence is along the first curvature vector with respect to the enveloping space.

Proceeding as above we can deduce the following theorem for the secondary K_λ curve.

Theorem (4.2)—A secondary K_λ curve is a union curve if it is a geodesic of the subspace and in this case the congruence is along the first curvature vector with respect to the enveloping space.

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