FOURFOLD GEOMETRY IN AN INFINITESIMAL DOMAIN OF HILBERT SPACE

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This paper gives a brief exposition of an infinitesimal fourfold geometry in a linear subspace generated by a set of infinitesimal vectors, each having an infinite number of components in a real Hilbert space. At a particular point of the fourfold besides the tangent subspace of four dimensions there are successive normal subspaces of 10, 20, 35 and generally of $\binom{4+K}{K+1}$ dimensions having characteristic features. Based on the tangent subspace only with fundamental tensors of the second rank defined in it, it is found that the processes of ordinary tensor calculus are applicable in the infinitesimal fourfold. On the normal subspaces, however, using a system of abridged notation a more extended form of tensor calculus has been evolved.

1. Parametric Equations of the Fourfold and the Transformation Scheme

Let

$$\rho = \sum_{i=1}^{m} S_i \alpha^i \qquad \ldots \quad (1.1)$$

represent an m-dimensional linear subspace generated by a system of m infinitesimal orthonormal vectors α^i passing through the origin in a real Hilbert space characterized by an infinity of mutually orthogonal coordinate axes. We have then (Ghosh 1970) the scalar product

$$\begin{Bmatrix} \alpha^k \\ \alpha^l \end{Bmatrix} = \int_a^b f^k(x) f^l(x)\, dx \begin{matrix} = 1 & \text{if } k = l \\ = 0 & \text{if } k \neq l \end{matrix} \Bigg\} \quad k, l = 1, 2, \ldots m. \qquad \ldots \quad (1.2)$$

If now the m-scalar coefficients S_i in (1.1) are analytic functions of four variables v^0, v^1, v^2 and v^3 in a given region we define a fourfold immersed in the m-dimensional domain of the Hilbert space, which we call shortly H-space by means of the parametric equations

$$S_i = \sigma_i(v^0, v^1, v^2, v^3) \quad i = 1, 2, \ldots m. \qquad \ldots \quad (1.3)$$

Fixing upon a point P on the fourfold corresponding to definite values v^λ let us form the H-vector

$$C = \sum_{i=1}^{m} \sigma_i(v^\lambda) \alpha^i. \qquad \ldots \quad (1.4)$$

The tangent subspace I_0 of the fourfold at P is then spanned by the four H-vectors

$$C_\lambda = \sum_{i=1}^{m} \frac{\partial \sigma_i}{\partial v^\lambda} \alpha^i \quad \lambda = 0, 1, 2, 3. \qquad \dots \quad \dots \quad \dots \quad (1.5)$$

The osculating subspace I_{01} of the second order at P is spanned by the H-vectors C_λ and $C_{\lambda_1\lambda_2}$, where

$$C_{\lambda_1\lambda_2} = \sum_{i=1}^{m} \frac{\partial^2 \sigma_i}{\partial v^{\lambda_1} \partial v^{\lambda_2}} \alpha^i \qquad \dots \quad \dots \quad \dots \quad (1.6)$$

and so on for higher order osculating subspaces.

Let us next introduce a general transformation scheme from the system v^λ to the system v'^λ by means of the set of reversible transformation equations

$$v'^\lambda = \phi^\lambda(v^0, v^1, v^2, v^3) \quad \lambda = 0, 1, 2, 3. \qquad \dots \quad \dots \quad (1.7)$$

The equations of transformation for C_λ and $C_{\lambda_1\lambda_2}$ in the v-space are then

$$C'_\mu = C_\lambda \frac{\partial v^\lambda}{\partial v'^\mu} \qquad \dots \quad \dots \quad \dots \quad \dots \quad (1.8)$$

$$C'_{\mu_1\mu_2} = C_{\lambda_1\lambda_2} \frac{\partial v^{\lambda_1}}{\partial v'^{\mu_1}} \cdot \frac{\partial v^{\lambda_2}}{\partial v'^{\mu_2}} + C_\lambda \frac{\partial^2 v^\lambda}{\partial v'^{\mu_1} \partial v'^{\mu_2}}. \qquad \dots \quad \dots \quad (1.9)$$

2. Fundamental Tensors in Tangent Subspace

Consider now the symmetrical determinant of order 4 represented in the compact notation

$$\Delta_1 = \begin{Bmatrix} C_0 C_1 C_2 C_3 \\ C_0 C_1 C_2 C_3 \end{Bmatrix} \qquad \dots \quad \dots \quad \dots \quad \dots \quad (2.1)$$

which yields the $(\lambda-\mu)$th element of the determinant as the scalar product of the vectors C_λ and C_μ chosen respectively from the first and second rows of Δ_1.

From (1.5) the value of the scalar product

$$\begin{Bmatrix} C_\lambda \\ C_\mu \end{Bmatrix} = \sum_{i=1}^{m} \frac{\partial \sigma_i}{\partial v^\lambda} \cdot \frac{\partial \sigma_i}{\partial v^\mu}. \qquad \dots \quad \dots \quad \dots \quad (2.2)$$

We have then $\begin{Bmatrix} C_\lambda \\ C_\mu \end{Bmatrix}$ as the fundamental covariant tensor in the v-space obeying the transformation law

$$\begin{Bmatrix} C'_\nu \\ C'_\sigma \end{Bmatrix} = \begin{Bmatrix} C_\lambda \\ C_\mu \end{Bmatrix} \frac{\partial v^\lambda}{\partial v'^\nu} \cdot \frac{\partial v^\mu}{\partial v'^\sigma}$$

the line-element being given by

$$ds^2 = \begin{Bmatrix} C_\lambda \\ C_\mu \end{Bmatrix} dv^\lambda dv^\mu. \qquad \dots \quad \dots \quad \dots \quad \dots \quad (2.3)$$

Let us denote the associated contravariant vector to C_λ by C^μ, then since

$$\begin{Bmatrix} C_\lambda \\ C_\mu \end{Bmatrix}\begin{Bmatrix} C^\mu \\ C^\nu \end{Bmatrix} = \delta^\lambda_\nu$$

the contravariant fundamental tensor $\begin{Bmatrix} C^\mu \\ C^\nu \end{Bmatrix}$ is the cofactor of $\begin{Bmatrix} C_\mu \\ C_\nu \end{Bmatrix}$ in the determinant Δ_1, divided by Δ_1. Thus

$$C^\mu = C_\lambda \begin{Bmatrix} C^\lambda \\ C^\mu \end{Bmatrix} \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.4)$$

the transformation formula being

$$C'^\nu = C^\lambda \frac{\partial v'^\nu}{\partial v^\lambda}. \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.5)$$

From (1.9) one obtains the relation

$$C'_{\mu_1\mu_2} - C'_\mu \begin{Bmatrix} C'^\mu \\ C'_{\mu_1\mu_2} \end{Bmatrix} = \left(C_{\lambda_1\lambda_2} - C_\lambda \begin{Bmatrix} C^\lambda \\ C_{\lambda_1\lambda_2} \end{Bmatrix} \right) \frac{\partial v^{\lambda_1}}{\partial v'^{\mu_1}} \cdot \frac{\partial v^{\lambda_2}}{\partial v'^{\mu_2}} \quad \dots \quad (2.6)$$

showing that

$$\overset{*}{C}_{\lambda_1\lambda_2} = C_{\lambda_1\lambda_2} - C_\lambda \begin{Bmatrix} C^\lambda \\ C_{\lambda_1\lambda_2} \end{Bmatrix} \quad \dots \quad \dots \quad \dots \quad (2.7)$$

is a symmetric covariant tensor in the v-space.

Further the system of H-vectors $\overset{*}{C}_{\lambda_1\lambda_2}$ is orthogonal to the tangent subspace, one can also show that

$$\left(\overset{*}{C}_{\lambda_1\lambda_2}\right)^*_{\lambda_3} = \frac{\partial}{\partial v^{\lambda_3}} \overset{*}{C}_{\lambda_1\lambda_2} - \overset{*}{C}_{\lambda\lambda_2} \begin{Bmatrix} C^\lambda \\ C_{\lambda_1\lambda_3} \end{Bmatrix} - \overset{*}{C}_{\lambda_1\lambda} \begin{Bmatrix} C^\lambda \\ C_{\lambda_3\lambda_2} \end{Bmatrix} \quad \dots \quad (2.8)$$

is a third rank covariant tensor in the v-space.

In the above the quantity $\begin{Bmatrix} C^\lambda \\ C_{\lambda_1\lambda_2} \end{Bmatrix}$ is expressible in a more explicit form

$$\begin{Bmatrix} C^\lambda \\ C^\sigma \end{Bmatrix}\begin{Bmatrix} C_\sigma \\ C_{\lambda_1\lambda_2} \end{Bmatrix} = \begin{pmatrix} C_\lambda \\ C_{\lambda_1\lambda_2} \end{pmatrix} \Delta_1 / \Delta_1 \quad \dots \quad \dots \quad \dots \quad (2.9)$$

where the symbol $\begin{pmatrix} C_\lambda \\ C_{\lambda_1\lambda_2} \end{pmatrix}\Delta_1$ denotes the scalar determinant (2.1) in which C_λ in the first row is replaced by $C_{\lambda_1\lambda_2}$.

It can be verified that

$$\begin{Bmatrix} C^\lambda \\ C_{\lambda_1\lambda_2} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} C^\lambda \\ C^\sigma \end{Bmatrix} \left(\partial_{\lambda_1} \begin{Bmatrix} C_{\lambda_2} \\ C_\sigma \end{Bmatrix} + \partial_{\lambda_2} \begin{Bmatrix} C_{\lambda_1} \\ C_\sigma \end{Bmatrix} - \partial_\sigma \begin{Bmatrix} C_{\lambda_1} \\ C_{\lambda_2} \end{Bmatrix} \right). \quad \dots \quad (2.10)$$

Hence $\begin{Bmatrix} C^\lambda \\ C_{\lambda_1\lambda_2} \end{Bmatrix}$ denotes the Christoffel symbol of the second kind in the v-space.

Differentiating (2.5) and making use of (1.9) it follows that the H-vector

$$(C^\lambda)^*_\nu = \frac{\partial}{\partial v^\nu} C^\lambda + C^\mu \begin{Bmatrix} C^\lambda \\ C_{\mu\nu} \end{Bmatrix} \quad \dots \quad \dots \quad \dots \quad (2.11)$$

obeys the transformation law of a mixed tensor of the second rank in the v-space.

From (2.11) we obtain the relation

$$\begin{Bmatrix} (C^{\lambda})^*_{\lambda_3} \\ C^*_{\lambda_1\lambda_2} \end{Bmatrix} - \begin{Bmatrix} (C^{\lambda})^*_{\lambda_2} \\ C^*_{\lambda_1\lambda_3} \end{Bmatrix} = \frac{\partial}{\partial v^{\lambda_3}} \begin{Bmatrix} C^{\lambda} \\ C_{\lambda_1\lambda_2} \end{Bmatrix} - \frac{\partial}{\partial v^{\lambda_2}} \begin{Bmatrix} C^{\lambda} \\ C_{\lambda_1\lambda_3} \end{Bmatrix}$$

$$+ \begin{Bmatrix} C^{\mu} \\ C_{\lambda_1\lambda_2} \end{Bmatrix} \begin{Bmatrix} C^{\lambda} \\ C_{\mu\lambda_3} \end{Bmatrix} - \begin{Bmatrix} C^{\mu} \\ C_{\lambda_1\lambda_3} \end{Bmatrix} \begin{Bmatrix} C^{\lambda} \\ C_{\mu\lambda_2} \end{Bmatrix} = R^{\lambda}_{\lambda_1\lambda_2\lambda_3}. \quad .. \text{(2.12)}$$

It can also be shown that

$$\left(C^*_{\lambda_1\lambda_3}\right)^*_{\lambda_2} - \left(C^*_{\lambda_1\lambda_2}\right)^*_{\lambda_3} = C_{\lambda} R^{\lambda}_{\lambda_1\lambda_2\lambda_3}. \quad .. \quad .. \quad .. \text{(2.13)}$$

$R^{\lambda}_{\lambda_1\lambda_2\lambda_3}$ is the Riemann curvature tensor of the fourth rank in the v-space.

3. Normal Subspaces of the Fourfold

The system of 10 mutually independent H-vectors $C^*_{\lambda_1\lambda_2}$ defined as a covariant tensor in (2.6) forms the first normal subspace I_1 of the fourfold at P. This is classified into four vectors of the type $C^*_{\sigma_1\sigma_1}$ and six of the type $C^*_{\sigma_1\sigma_2}$ $(\sigma_1 < \sigma_2)$. Consider now the non-vanishing scalar determinant of the tenth order

$$\Delta^*_2 = \begin{Bmatrix} C^*_{00} C^*_{01} \dots C^*_{23} C^*_{33} \\ C^*_{00} C^*_{01} \dots C^*_{23} C^*_{33} \end{Bmatrix}. \quad .. \quad .. \quad .. \text{(3.1)}$$

We introduce 10 contravariant H-vectors $C^{*\lambda_1\lambda_2}$ $(\lambda_1 \leqslant \lambda_2)$ defined by means of the equation

$$C^{*\lambda_1\lambda_2} = \sum C^*_{\sigma_1\sigma_2} \begin{Bmatrix} C^*_{\sigma_1\sigma_2} \\ C^*_{\lambda_1\lambda_2} \end{Bmatrix}_{\mu} \Bigg/ \Delta^*_2 \quad .. \quad .. \quad .. \text{(3.2)}$$

where the summation indices are restricted by the condition $\sigma_1 \leqslant \sigma_2 = 0, 1, 2, 3$ and the suffix μ attached to the element of Δ^*_2 indicates that its cofactor in the determinant is to be taken. Elsewhere (Ghosh 1940) this system of vectors has been called 'conjugate'.

From (3.2) we obtain

$$\begin{Bmatrix} C^{*\lambda_1\lambda_2} \\ C^*_{\nu_1\nu_2} \end{Bmatrix} = \delta^{\lambda_1}_{\nu_1} \delta^{\lambda_2}_{\nu_2}$$

$$C^{*\lambda_1\lambda_2} = C^*_{\sigma_1\sigma_2} \begin{Bmatrix} C^{*\sigma_1\sigma_2} \\ C^{*\lambda_1\lambda_2} \end{Bmatrix}, \quad C^*_{\sigma_1\sigma_2} = C^{*\lambda_1\lambda_2} \begin{Bmatrix} C^*_{\sigma_1\sigma_2} \\ C^*_{\lambda_1\lambda_2} \end{Bmatrix}. \quad .. \text{(3.3)}$$

It may be noted that the transformation formula

$$C'^{*}_{\mu_1\mu_2} = C^{*}_{\lambda_1\lambda_2}\frac{\partial v^{\lambda_1}}{\partial v'^{\mu_1}}\cdot\frac{\partial v^{\lambda_2}}{\partial v'^{\mu_2}}$$

becomes

$$C'^{*}_{\rho_1\rho_2} = C^{*}_{\sigma_1\sigma_1}\frac{\partial v^{\sigma_1}}{\partial v'^{\rho_1}}\cdot\frac{\partial v^{\sigma_1}}{\partial v'^{\rho_2}}+C^{*}_{\sigma_1\sigma_2}\left(\frac{\partial v^{\sigma_1}}{\partial v'^{\rho_1}}\cdot\frac{\partial v^{\sigma_2}}{\partial v'^{\rho_2}}+\frac{\partial v^{\sigma_2}}{\partial v'^{\rho_1}}\cdot\frac{\partial v^{\sigma_1}}{\partial v'^{\rho_2}}\right) \quad \ldots \quad (3.4)$$

and the transformation formula

$$C^{*}_{\lambda_1\lambda_2} = C'^{*}_{\mu_1\mu_2}\frac{\partial v'^{\mu_1}}{\partial v^{\lambda_1}}\cdot\frac{\partial v'^{\mu_2}}{\partial v^{\lambda_2}}$$

becomes

$$C^{*}_{\rho_1\rho_2} = C'^{*}_{\sigma_1\sigma_1}\frac{\partial v'^{\sigma_1}}{\partial v^{\rho_1}}\cdot\frac{\partial v'^{\sigma_1}}{\partial v^{\rho_2}}+C'^{*}_{\sigma_1\sigma_2}\left(\frac{\partial v'^{\sigma_1}}{\partial v^{\rho_1}}\cdot\frac{\partial v'^{\sigma_2}}{\partial v^{\rho_2}}+\frac{\partial v'^{\sigma_2}}{\partial v^{\rho_1}}\cdot\frac{\partial v'^{\sigma_1}}{\partial v^{\rho_2}}\right) \quad \ldots \quad (3.5)$$

$$(\sigma_1 < \sigma_2,\ \rho_1 \leqslant \rho_2)$$

when written in terms of the 10 mutually independent components of $C^{*}_{\lambda_1\lambda_2}$ and $C'^{*}_{\mu_1\mu_2}$. The second normal subspace I_2 at the point P is then determined by the 20 mutually independent H-vectors

$$C^{*}_{\lambda_1\lambda_2\lambda_3} = C_{\lambda_1\lambda_2\lambda_3} - C^{*}_{\sigma_1\sigma_2}\left\{\begin{matrix} C^{*\sigma_1\sigma_2} \\ C_{\lambda_1\lambda_2\lambda_3} \end{matrix}\right\} - C_{\lambda}\left\{\begin{matrix} C^{\lambda} \\ C_{\lambda_1\lambda_2\lambda_3} \end{matrix}\right\} \quad \ldots \quad \ldots \quad (3.6)$$

each member of which is orthogonal to the osculating subspace I_{01}. In the v-space, with unrestricted λ's, $C^{*}_{\lambda_1\lambda_2\lambda_3}$ behaves as a covariant tensor of the third rank. The above process can readily be extended to determine higher order normal subspaces.

4. Double-script Vectors and Tensors

Let us now proceed to apply tensor methods to the geometry of the normal subspaces. We regard the 10 double-script H-vectors of the type $C^{*}_{\sigma_1\sigma_2}$ as forming the first normal linear subspace I_1. From (3.4) and (3.5) we write the transformation formulae as

$$C'^{*}_{\rho_1\rho_2} = C^{*}_{\sigma_1\sigma_2}\begin{pmatrix}\sigma_1\sigma_2 \\ \rho'_1\rho'_2\end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

and

$$C^{*}_{\rho_1\rho_2} = C'^{*}_{\sigma_1\sigma_2}\begin{pmatrix}\sigma'_1\sigma'_2 \\ \rho_1\rho_2\end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

where the coefficients of transformation obey the relation

$$\begin{pmatrix}\sigma_1\sigma_2 \\ \mu'_1\mu'_2\end{pmatrix}\begin{pmatrix}\mu'_1\mu'_2 \\ \rho_1\rho_2\end{pmatrix} = \delta^{\sigma_1}_{\rho_1}\delta^{\sigma_2}_{\rho_2}. \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

The summation indices being restricted by the condition

$$\mu_1' \leqslant \mu_2' = 0, 1, 2, 3.$$

The transformation formulae of the double-script contravariant H-vectors defined in (3.2) are given by

$$C'^{*\sigma_1\sigma_2} = C^{*\rho_1\rho_2}\binom{\sigma_1'\sigma_2'}{\rho_1\rho_2} \quad \ldots \quad (4.4)$$

$$C^{*\sigma_1\sigma_2} = C'^{*\rho_1\rho_2}\binom{\sigma_1\sigma_2}{\rho_1'\rho_2'}. \quad \ldots \quad (4.5)$$

Since the product vector $dv^{\sigma_1}dv^{\sigma_2}$ transforms in accordance with (4.4), it follows from (4.1) that

$$\begin{Bmatrix} C^*_{\sigma_1\sigma_2} \\ C^*_{\rho_1\rho_2} \end{Bmatrix}(dv^{\sigma_1}dv^{\sigma_2})(dv^{\rho_1}dv^{\rho_2}) \quad \ldots \quad (4.6)$$

is the fundamental invariant in the 10 dimensional normal space I_1. From (4.1) we obtain the transformation formula

$$\begin{Bmatrix} C'^*_{\mu_1\mu_2} \\ C'^*_{\lambda_1\lambda_2} \end{Bmatrix} = \begin{Bmatrix} C^*_{\sigma_1\sigma_2} \\ C^*_{\rho_1\rho_2} \end{Bmatrix}\binom{\sigma_1\sigma_2}{\mu_1'\mu_2'}\binom{\rho_1\rho_2}{\lambda_1'\lambda_2'}. \quad \ldots \quad (4.7)$$

Starting from the differential operator

$$\frac{\partial}{\partial v'^{\mu}} = \frac{\partial v^{\lambda}}{\partial v'^{\mu}}\cdot\frac{\partial}{\partial v^{\lambda}}$$

one can arrive at the double-script differential operator

$$\frac{\partial^2}{\partial v'^{\mu_1}\partial v'^{\mu_2}} - \begin{Bmatrix} C'_{\mu_1\mu_2} \\ C'^{\mu} \end{Bmatrix}\frac{\partial}{\partial v'^{\mu}} = \binom{\lambda_1\lambda_2}{\mu_1'\mu_2'}\left[\frac{\partial^2}{\partial v^{\lambda_1}\partial v^{\lambda_2}} - \begin{Bmatrix} C_{\lambda_1\lambda_2} \\ C^{\lambda} \end{Bmatrix}\frac{\partial}{\partial v^{\lambda}}\right]. \quad \ldots \quad (4.8)$$

Expressing the above as

$$\partial'^*_{\mu_1\mu_2} = \binom{\lambda_1\lambda_2}{\mu_1'\mu_2'}\partial^*_{\lambda_1\lambda_2} \quad \ldots \quad (4.9)$$

and operating on

$$C'^*_{\rho_1\rho_2} = C^*_{\sigma_1\sigma_2}\binom{\sigma_1\sigma_2}{\rho_1'\rho_2'}$$

one obtains the transformation equation

$$\partial'^*_{\mu_1\mu_2}C'^*_{\rho_1\rho_2} - C'^*_{\mu\rho}\begin{Bmatrix} \partial'^*_{\mu_1\mu_2}C'^*_{\rho_1\rho_2} \\ C'^{*\mu\rho} \end{Bmatrix} = \binom{\lambda_1\lambda_2}{\mu_1'\mu_2'}\binom{\sigma_1\sigma_2}{\rho_1'\rho_2'}\left[\partial^*_{\lambda_1\lambda_2}C^*_{\sigma_1\sigma_2} - C^*_{\lambda\sigma}\begin{Bmatrix} \partial^*_{\lambda_1\lambda_2}C^*_{\sigma_1\sigma_2} \\ C^{*\lambda\sigma} \end{Bmatrix}\right] \quad \ldots \quad (4.10)$$

which is analogous to (2.6). Other tensor formulae noted before may thus be extended. In the other higher normal subspaces similar extensions may be performed, thus building up what may be called the multiple-script tensor calculus.

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