ON DISTURBANCES IN A VISCO-ELASTIC MEDIUM OF VOIGT TYPE IN CONTACT WITH A LIQUID MEDIUM AND SUBJECTED TO A MAGNETIC FIELD

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(Communicated by B. Sen, F.N.I.)

(*Received 12 May 1969*)

The investigation of the problem of disturbances in a visco-elastic medium of Voigt type in contact with a liquid medium acted upon by a magnetic field forms the chief interest of this note. The electromagnetic equations of Maxwell, equation of elasticity and the stress-strain relations of Voigt material have been made use of to solve the problem.

INTRODUCTION

It is only in recent years that the mechanical phenomena are being assessed in the presence of magnetic field and out of such investigations have emerged the interdisciplinary subjects such as magneto-hydrodynamics and magneto-elasticity. These studies have relevance to a large number of geophysical and seismological problems as evidenced by the works of Knopoff (1955), Rikitake (1952), Yukutake (1957) and Bakshi (1968). These deal with the interaction of elastic and magnetic fields but the visco-elastic materials may well afford to be a representation of the interior of the earth. As such, the studies relating to the effect of magnetic field on visco-elastic materials seem to be worth attempting (*see* Sinha (*in press*), Das (1967), Bakshi (1968) and Roy Chowdhury (1969)). But as far as the present author is aware, there is only one attempt by Nowacki and Kaliski (1963), and an attempt by the author (Basu Mallik 1969) which treat of the effect of magnetic field on visco-elastic material in contact with a liquid. This note is an effort of a similar kind and it is concerned with the investigation of disturbances in a visco-elastic medium of Voigt type in contact with a liquid medium and acted upon by a magnetic field.

PROBLEM, FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITION

We consider two perfectly conductive media, a visco-elastic solid and an ideal liquid, which are permeated by a primary magnetic field of intensity H, normal to their plane of contact. Let us apply a transient force, $P_0e^{-\alpha t}$ ($\alpha > 0$), where P_0 is a constant along this plane. This force when interacting with the magnetic field produces a coupling of the electromagnetic and mechanical fields.

The origin is chosen at the contact plane with the z-axis directed normally to the plane. The region $z > 0$ is occupied by the visco-elastic medium and the region $z < 0$ by the liquid.

The problem is to investigate the nature of disturbances in the solid subject to the mechanical as well as assigned initial magnetic field and the influence of magnetic field on the disturbances.

Hence, the equation representing the interaction between the mechanical and electromagnetic fields constitutes the fundamental equations of the problem.

The electromagnetic equations of a perfect conductor, as given by Nowacki and Kaliski (1963), are

$$\left.\begin{aligned} \operatorname{rot} \vec{h} &= \frac{4\pi}{c} \vec{J} \\ \operatorname{rot} \vec{E} &= -\frac{1}{c} \frac{\delta \vec{h}}{\delta t} \\ \vec{E}+\frac{1}{c}\left(\frac{\delta \vec{u}}{\delta t} \times \vec{H}\right) &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $\vec{J}$ is the current density vector; $\vec{h}$, the perturbation of the initial magnetic intensity vector; $\vec{u}$, the mechanical displacement vector; and $\vec{E}$, the electric field intensity vector. The magnetic permeability μ is taken to be unity for reasons of simplicity.

The equation representing the mechanical field is given by

$$\rho_1 \frac{\delta^2 \vec{u}}{\delta t^2} = G \nabla^2 \vec{u} + (\lambda + G) \operatorname{grad} \operatorname{div} \vec{u} + \frac{1}{c} (\vec{J} \times \vec{H}) \quad \ldots \quad \ldots \quad (2)$$

where λ and G are given by

$$\lambda = \lambda_1 + \lambda_2 \delta/\delta t; \quad G = G_1 + G_2 \delta/\delta t$$

λ_1, G_1, λ_2 and G_2 are material constants and ρ_1 is the density of visco-elastic medium.

If we assume the component of displacement in the z-direction as u and the magnetic field in the x-direction, the equation of motion in z-direction, eliminating $\vec{h}$, $\vec{J}$, $\vec{E}$ in terms of $\vec{u}$ from eqns. (1) and (2), may thus be obtained as

$$\frac{1}{\rho_1}\left[(\lambda_1 + 2G_1) + (\lambda_2 + 2G_2)\delta/\delta t + \frac{H^2}{4\pi}\right] \frac{\delta^2 u}{\delta z^2} - \frac{\delta^2 u}{\delta t^2} = 0$$

which takes the form

$$(l_1+m_1\delta/\delta t)\frac{\delta^2 u}{\delta z^2}-\frac{\delta^2 u}{\delta t^2}=0 \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

where

$$l_1=\frac{1}{\rho_1}\left(\lambda_1+2G_1+\frac{H^2}{4\pi}\right)$$

and

$$m_1=\frac{1}{\rho_1}(\lambda_2+2G_2).$$

Therefore, the fundamental equations for the coupling are

$$(l_1+m_1\delta/\delta t)\frac{\delta^2 u}{\delta z^2}-\frac{\delta^2 u}{\delta t^2}=0 \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$h=H\frac{\delta u}{\delta z}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

The boundary conditions which will be treated for the problem are as follows:

(i) The total stress (composed of electromagnetic and elastic stress) is continuous on $z=0$.

(ii) The tangential component of electric and magnetic fields is continuous on $z=0$.

(iii) The displacement is continuous on $z=0$.

In accordance with our assumptions the symbols

$$u=u^{(2)},\quad h=h^{(2)},\quad a_2^2={}_2a_0^2=\frac{H^2}{4\pi\rho_2}$$

are used for the liquid and the symbols

$$u=u^{(1)},\quad h=h^{(1)},\quad a^2={}_1a_0^2+c^2,\quad {}_1a_0^2=\frac{H^2}{4\pi\rho_1},\quad c^2=\frac{\mu}{\rho_1}$$

are used for the solid.

As given by Nowacki and Kaliski (1963), we set two variants of the boundary conditions, i.e.
either

$$u^{(1)}=u^{(2)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\rho_1 a^2\frac{\delta u^{(1)}}{\delta z}-\rho_2 a_2^2\frac{\delta u^{(2)}}{\delta z}=P_0 e^{-\alpha t} \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

or

$$h^{(1)}=h^{(2)},\quad \text{i.e. } \frac{\delta u^{(1)}}{\delta z}=\frac{\delta u^{(2)}}{\delta z} \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$\rho_1 c^2\frac{\delta u^{(1)}}{\delta z}=P_0 e^{-\alpha t}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

The suffixes 1 and 2 are denoted for the solid and liquid respectively.

Solution

As is customary, let us apply the method of transform. The Laplace transform $\overline{f(p)}$ of $f(t)$ is defined by

$$\overline{f(p)} = \int_0^\infty f(t)e^{-pt}\,dt \quad (p > 0).$$

Taking the Laplace transform of eqn. (4), we get

$$\frac{\delta^2 \bar{u}}{\delta z^2} = \frac{p^2}{l_1 + m_1 p}\,\bar{u}$$

the solution of which, subject to the appropriate condition, is

$$\bar{u}^{(1)} = A_1 \exp\left(-\frac{p}{\sqrt{l_1+m_1p}}\,z\right) \quad \ldots \quad (10)$$

where A_1 is a constant.

As given by Nowacki and Kaliski (1963), we have

$$\bar{u}^{(2)} = A_2 \exp\,(p/a_2)z \quad \ldots \quad (11)$$

where A_2 is a constant.

Applying the boundary conditions (6) and (7) to the above solutions (4) and (5), the values of the constants A_1 and A_2 may be obtained which when put in (10) give

$$\bar{u}^{(1)} = \frac{P_0 \exp\left(\frac{-p}{\sqrt{l_1+m_1p}}\right)z}{(p+\alpha)\left[\frac{\rho_1 a^2 p}{\sqrt{l_1+m_1p}} + \rho_2 a_2 p\right]}. \quad \ldots \quad (12)$$

Applying again the boundary conditions (8) and (9) to the solutions (4) and (5) another set of values of the constants A_1 and A_2 may be obtained which when put in (10) give another value of $\bar{u}^{(1)}$

$$\bar{u}^{(1)} = \frac{P_0\sqrt{l_1+m_1p}\,\exp\left(\frac{-p}{\sqrt{l_1+m_1p}}\right)z}{\rho_1 c^2 (p+\alpha)}. \quad \ldots \quad (13)$$

Now we have to find the inverse transform and, to do so, let us now use the well-known notation due to Vander Pol and Bremmer. Thus, we can write eqn. (13) as

$$u_1(z,p) \doteqdot \frac{P_0}{\rho_1 c^2}\,\frac{\sqrt{m_1}\sqrt{p+\beta}}{(p+\alpha)}\,\exp\left(\frac{-kp}{\sqrt{p+\beta}}\right) \quad \ldots \quad (14)$$

where

$$k = \frac{z}{\sqrt{m_1}} \quad \text{and} \quad \beta = \sqrt{\frac{l_1}{m_1}}.$$

Writing p for $p+\beta$, we find

$$u_1(z,t)e^{\beta t} \doteqdot \frac{P_0}{\rho_1 c^2}\,\frac{\sqrt{m_1}p}{(p-\beta)(p+\alpha-\beta)}\,\frac{\exp\left(\frac{k\beta}{\sqrt{p}}\right)}{\sqrt{p}}\,\exp\left(\frac{-k}{\sqrt{p}}\right).$$

Now

$$\frac{\exp\left(\dfrac{k\beta}{\sqrt{p}}\right)}{\sqrt{p}} = \sum_{n=0}^{\infty} \frac{k^n\beta^n}{n!\,(\sqrt{p})^{n+1}}.$$

Therefore

$$u_1(z,t)e^{\beta t} \risingdotseq \frac{\sqrt{m_1}\,P_0}{\rho_1c^2}\,\frac{p}{(p-\beta)(p+\alpha-\beta)}\sum_{n=0}^{\infty}\frac{k^n\beta^n}{n!}\,\frac{\exp\left(\dfrac{-k}{\sqrt{p}}\right)}{(\sqrt{p})^{n+1}}. \quad \ldots \quad (15)$$

Now

$$\frac{\exp\left(\dfrac{-k}{\sqrt{p}}\right)}{(\sqrt{p})^{n+1}} \risingdotseq \frac{1}{\sqrt{\pi}}\,2^{n/2}t^{\frac{n-1}{2}}e^{\frac{-k^2}{8t}}D_{-n}\left(\frac{k}{\sqrt{2t}}\right)$$

where D_{-n} is the parabolic cylindrical function of negative integral transform (*vide* Erdelyi 1946)
and

$$\frac{p}{(p-\beta)(p+\alpha-\beta)} \risingdotseq \left|\begin{array}{ll}\dfrac{1}{\alpha}\{\beta e^{\beta t}-(\beta-\alpha)e^{(\beta-\alpha)t}\} & \text{for } \alpha > \beta \\ \dfrac{1}{\alpha}\{\beta e^{-\beta t}+(\beta-\alpha)e^{(\alpha-\beta)t}\} & \text{for } \alpha < \beta\end{array}\right| = f(t)\ (\text{say})$$

and

$$\frac{k^n\beta^n}{n!\,(\sqrt{p})^{n+1}}\,e^{-k\sqrt{p}} \risingdotseq \sum_{n=0}^{\infty}\frac{1}{\sqrt{\pi}}\,2^{n/2}t^{\frac{n-1}{2}}e^{\frac{-k^2}{8t}}D_{-n}\left(\frac{k}{\sqrt{2t}}\right)\frac{k^n\beta^n}{n!} = g(t)\ (\text{say}).$$

Hence, applying the convolution theorem on (15), we find

$$u_1(z,t)e^{\beta t} = \frac{P_0\sqrt{m_1}}{\rho_1c^2}\int_0^t g(\tau)f(t-\tau)\,d\tau.$$

Therefore

$$u_1(z,t) = \frac{P_0\sqrt{m_1}}{\rho_1c^2}\,e^{-\beta t}K(t) \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

where

$$K(t) = \int_0^t g(\tau)f(t-\tau)\,d\tau.$$

The determination of $u_1(z, t)$ needs numerical computations but, however, it exhibits transient characteristics. It is also evident that the decay factor β contains H of the magnetic field and thus a magnetic field accentuates the decaying of the displacement.

Taking for the other case, i.e. from the boundary conditions (8) and (9), we proceed as before. We can, thus, write

$$u_1(z,t) \risingdotseq \frac{P_0\sqrt{m_1}}{p(p+\alpha)}\sqrt{p+\beta}\,e^{\frac{-kp}{\sqrt{k+\beta}}} \times \frac{1}{\rho_2a_2\sqrt{m_1}} \times \frac{1}{\sqrt{m_1}\dfrac{\rho_1a^2}{\rho_2a_2}+\sqrt{p+\beta}} \qquad (17)$$

which can be written as

$$u_1(z,\, t)e^{\beta t} \doteqdot \frac{P_0 p}{(p-\beta)(p+\alpha-\beta)}\, \frac{e^{\frac{k\beta}{\sqrt{p}}}}{\sqrt{p}}\, e^{-k\sqrt{p}} \times \frac{1}{\rho_2 a_2} \times \frac{1}{\frac{\alpha}{\sqrt{m_1}}+\sqrt{p}}$$

where

$$\alpha = \frac{\rho_1 a^2}{\rho_2 a_2}.$$

Therefore

$$u_1(z,\, t)e^{\beta t} \doteqdot \frac{P_0}{\rho_2 a_2} f_2(p) \times f_1(p) \text{ (say)} \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

where we take

$$f_1(p) = \frac{1}{\frac{\alpha}{\sqrt{m_1}}+\sqrt{p}}.$$

Thus

$$f_1(z,\, t) \doteqdot \left[\frac{1}{\sqrt{\pi t}} - \frac{\alpha}{\sqrt{m_1}}\, e^{\frac{\alpha^2}{m_1}t}\right] \operatorname{erfc}\,(\alpha m_1^{-\frac{1}{2}} t^{\frac{1}{2}}) = h(t) \text{ (say)}.$$

Hence, applying again the convolution theorem on (18), we find

$$u_1(z,\, t)e^{\beta t} \doteqdot \frac{P_0}{\rho_2 a_2}\int_0^t K(\tau)h(t-\tau)\, d\tau.$$

Therefore

$$u_1(z,\, t) \doteqdot \frac{P_0 e^{-\beta t}}{\rho_2 a_2}\int_0^t K(\tau)h(t-\tau)\, d\tau \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

which also brings out the transient behaviour of the displacement in this case, too. A similar conclusion in regard to the magnetic field, as in the first case, holds true.

Acknowledgements

The author is thankful to Dr. D. K. Sinha for his kind help and guidance and also to Prof. S. Deb, F.N.I., and Prof. B. Mukherjee of the Department of Geological Sciences, Jadavpur University, for their keen interest in the work.

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