SEMILOCAL q-RINGS*

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A ring R with unity is said to be a right q-ring if every right ideal of R is quasi-injective as a right R-module. q-rings were introduced and characterized by Jain *et al.* (1969). The object of this paper is to study semilocal right q-rings. Besides other results we prove the following theorem: If R is a semilocal right q-ring and gRg is not a division ring for every idempotent g of R, then R is a finite direct sum of local right self-injective duo rings.

Throughout this paper we assume that every ring has unity element. If X is a subset of a ring R, $r(X)$ [resp. $l(X)$] will denote the right (resp. the left) annihilator of X in R. J will denote the Jacobson radical of a ring R. A ring R is said to be bound to its radical if $l(J) \cap r(J) \subset J$. A ring R is called local (resp. semilocal) if R/J is a division ring (resp. semi-simple artinian). It is known that in a semilocal ring R, the right (resp. left) socle of R is $l(J)$ [resp. $r(J)$]. A ring R is said to be a right (resp. left) duo ring if every right (resp. left) ideal in R is two-sided. R is said to be a duo ring if every one-sided ideal in R is two-sided. A right PF-ring R was characterized by Utumi (1967) as a right self-injective ring in which R/J is artinian and every non-zero right ideal contains a minimal right ideal. An ideal A of a ring R is called a mini ideal if A is minimal as a right ideal and as a left ideal in R. If N is a submodule of the right R-module M, then $N \subset{}' M$ will denote the fact that N is large in M.

§ 1. In this section we generalize a theorem of Brown and McCoy (1960). This will reduce our problem on semilocal rings to the case when R is bound to its radical. Besides, we record some results obtained by Jain *et al.* (1969) and Mohamed (*in press*) for easy reference.

Right q-rings were characterized by the following (Jain *et al.* 1969):

Theorem 1.1—The following are equivalent:

(1) R is a right q-ring;

(2) R is right self-injective and every right ideal in R is of the form eI, where e is an idempotent and I is a two-sided ideal;

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(3) R is right self-injective and every large right ideal in R is two-sided.

Lemma 1.2—A right self-injective right duo ring R is a duo ring.

PROOF: It is enough to prove that every principal left ideal in R is two-sided. Since R is right self-injective, $Ra = l(r(Ra))$, for every $a \in R$ (Ideka and Nakayama 1954, Theorem 1). Now, because R is a right duo ring, $r(Ra)$ is a two-sided ideal. Hence $l(r(Ra))$ is a two-sided ideal, completing the proof.

Corollary 1.3—A local right q-ring is a duo ring.

PROOF: Since R is local, the only idempotent in R is 1. Thus R is a right duo ring follows immediately by Theorem 1.1. Hence R is a duo ring by Lemma 1.2.

Lemma 1.4—Let R be a direct sum of a finite number of rings R_i. Then

(i) R is semilocal if and only if each R_i is,

(ii) R is bound to its radical if and only if each R_i is, and

(iii) R is a right q-ring if and only if each R_i is.

The proof is straightforward.

The following generalizes Theorem 7 of Brown and McCoy (1950).

Theorem 1.5—A semilocal ring R is a direct sum of two rings R_1 and R_2, where R_1 is semisimple artinian and R_2 is a semilocal ring bound to its radical.

PROOF: Let R_1 denote the maximal regular ideal of R. If $R_1 = 0$, then R is bound to its radical (Brown and McCoy 1950, Theorem 6). So, let $R_1 \neq 0$. We consider R_1 as a ring by itself and proceed to show that R_1 has a unity element. Since R/J is artinian, R does not contain an infinite number of orthogonal idempotents. Since R_1 is a regular ring, then by Theorem 2.1 of Kaplansky (1950) R_1 is artinian. Hence R_1 has a unity element as desired. Let e be the unity of R_1. Then e is a central idempotent in R. Let $R_2 = (1-e)R$. Then R is a direct sum of the rings R_1 and R_2. By Lemma 1.4, R_2 is a semilocal ring. Let $M(R_2)$ denote the maximal regular ideal of the ring R_2. Then, by Theorem 3 of Brown and McCoy (1950), $M(R_2) = R_1 \cap R_2 = 0$. Hence R_2 is bound to its radical (Brown and McCoy 1950, Theorem 6), which completes the proof.

The following two results were proved by Mohamed (*in press*):

Lemma 1.6—Every left ideal contained in the radical J of a right q-ring is two-sided.

Proposition 1.7—Let R be a right PF right q-ring which is bound to its radical. Then an indecomposable idempotent g is in the centre of R if and only if gRg is not a division ring.

We shall also have the occasion to use the following result of Utumi (1967).

Lemma 1.8—In a semilocal right self-injective ring $l(J) \subset r(J)$.

PROOF: (Sketched) by Lemma 4.1 of Utumi (1965), J coincides with the right singular ideal of R. Hence $xl(J) = 0$ for every $x \in J$. Therefore $l(J) \subset r(J)$.

§ 2. In this section R will denote a semilocal right q-ring which is bound to its radical. Since R is semilocal, $l(J)$ [resp. $r(J)$] is the right (resp. left) socle of R. Also, by Lemma 1.8, $l(J) \subset r(J)$ and then $l(J) \subset J$, because R is bound to J. This implies that J is a large right ideal of R. For if A is a right ideal of R such that $A \cap J = 0$, then $AJ = 0$, which implies that $A \subset l(J) \subset J$, and hence $A = 0$.

Lemma 2.1—Every right (or left) ideal of R containing J is two-sided.

PROOF: Since R is a right q-ring and J is a large right ideal of R, every right ideal of R containing J is two-sided by Theorem 1.1. Hence R/J is a right duo ring. But, since R/J is right self-injective, R/J is a duo ring by Lemma 1.2. In fact, R/J is a finite direct sum of division rings. This implies that every left ideal of R containing J is two-sided, completing the proof.

Proposition 2.2—Every minimal right ideal of R is a mini ideal and is of the form $el(J)$ for some indecomposable idempotent e of R. Conversely, if e is an indecomposable idempotent of R such that $el(J) \neq 0$, then $el(J)$ is a mini ideal of R.

PROOF: Let A be a minimal right ideal of R. Since $l(J) \subset r(J)$, $A \subset r(J)$. This implies that

$$l(A) \supset l(r(J)) \supset J.$$

Hence $l(A)$ is a two-sided ideal of R by 2.1.

Since R is right self-injective, $A \subset {}'eR$ for some idempotent e of R. Because A is a minimal right ideal, e is indecomposable. Now, $A \subset eR$ implies

$$A \subset {}'r(l(A)) \subset {}'eR$$

as right R-modules. Since $l(J)$ is the right socle of R, $l(J)$ is completely reducible as a right R-module. Hence $A \subset {}'r(l(A))$ implies

$$A \cap l(J) = r(l(A)) \cap l(J).$$

Therefore

$$A = A \cap l(J) = r(l(A)) \cap l(J)$$

proving that A is a two-sided ideal of R.

Now, $l(J) \subset J$ implies $A \subset J$. Hence, by Lemma 1.6, every left ideal contained in A is two-sided. This shows that A is a minimal left ideal, and hence a mini ideal.

Also, $A \subset {}'eR$ implies that

$$A = A \cap l(J) = eR \cap l(J) = el(J).$$

This proves the first statement of the proposition.

Conversely, let e be an indecomposable idempotent of R such that $el(J) \neq 0$. Since R is right self-injective, eR is an indecomposable injective right

R-module. Hence eR is a uniform right ideal of R. This implies that eR contains at most one minimal right ideal of R. Since $0 \neq el(J) \subset eR$ and $el(J)$ is a direct sum of minimal right ideals, it follows that $el(J)$ is a minimal right ideal of R. But then $el(J)$ is a mini ideal, completing the proof.

Lemma 2.3—Let B be a minimal left ideal of R. The following are equivalent:

$$(1)\ B \subset J, \qquad (2)\ B \text{ is two-sided}$$

$$(3)\ B \subset l(J), \quad (4)\ B \text{ is a mini ideal}$$

PROOF: (1) implies (2) by Lemma 1.6.

Assume (2). Let E be a large right ideal of R. Then $B \cap E \neq 0$. Since E is two-sided, by 1.1, $B \cap E$ is a non-zero left ideal contained in B. Then minimality of B implies that $B \subset E$. Since $l(J)$ is the intersection of all large right ideals, $B \subset l(J)$, proving (3).

Assume (3). Then, $l(J) \subset J$ implies $B \subset J$ which in turn implies that B is two-sided, by Lemma 1.6. Thus B is a direct sum of minimal right ideals. Since, by Proposition 2.2, every minimal right ideal is a mini ideal, B is a mini ideal. Hence (3) implies (4).

That (4) implies (1) follows immediately by the fact that every mini ideal of R is contained in $l(J)$.

The following is an immediate consequence.

Corollary 2.4—(i) $l(J) = J \cap r(J)$

(ii) $l(J)$ is large in $r(J)$ as right R-modules

In a right self-injective ring idempotents modulo the Jacobson radical can be lifted (cf. Utumi 1965). This implies that a right self-injective semilocal ring is semiperfect. Thus every non-zero idempotent of R is the sum of a finite number of orthogonal indecomposable idempotents.

Proposition 2.5—Suppose that $l(J) \subset' eR$ for some idempotent e of R. Let $e = e_1 + e_2 + \ldots + e_k$, where $\{e_i,\ 1 \leqslant i \leqslant k\}$ is a set of orthogonal indecomposable idempotents. Then

$$e_1 l(J),\ e_2 l(J), \ldots,\ e_k l(J)$$

are the only mini ideals of R.

PROOF: Since $l(J) \subset' eR$, $e_i l(J) = e_i R \cap l(J) \neq 0$, $1 \leqslant i \leqslant k$. Then, $e_i l(J)$ is a mini ideal, $1 \leqslant i \leqslant k$, by Proposition 2.2.

Now, let A be a mini ideal. Then

$$A = eA = e_1 A \oplus e_2 A \oplus \ldots \oplus e_k A.$$

Since A is a minimal right ideal, $A = e_s A$ for some $s \in \{1, 2, \ldots, k\}$. Thus $A = e_s A \subset e_s l(J)$. But then $A = e_s l(J)$, since $e_s l(J)$ is a mini ideal. This completes the proof.

Lemma 2.6—Let e and e' be orthogonal idempotents of R. Then $eRe' \subset l(J)$.

PROOF: For every $a \in R$, Ja is a two-sided ideal by Lemma 1.6. Hence

$$J . eRe' = JeR . e' = Jee' = 0.$$

This implies that $eRe' \subset r(J)$.

Also, by Lemma 2.1, R/J is a duo ring. Then

$$eR+J = Re+J$$

which implies that $eRe' \subset J$. Whence

$$eRe' \subset J \cap r(J).$$

The result then follows by Corollary 2.4.

Now, we prove the main result.

Theorem 2.8—If R is a semilocal right q-ring and gRg is not a division ring for every idempotent g of R, then R is a finite direct sum of local right self-injective duo rings.

PROOF: We first assert that R is bound to its radical. By Theorem 1.5, $R = R_1 \oplus R_2$, where R_1 is semisimple artinian and R_2 is a semilocal ring which is bound to its radical. If $R_1 \neq 0$, then R_1 contains minimal right ideals. Since R_1 is semiprime, every minimal right ideal is of the form gR_1 for some idempotent g of R_1 such that gR_1g is a division ring. But this is a contradiction, since $gR_1g = gRg$. Hence $R_1 = 0$ and R is bound to its radical, as asserted.

We complete the proof by considering two cases:

Case (i)—$l(J) = 0$. Let f be an idempotent of R. Then, by Lemma 2.6, $fR(1-f) \subset l(J) = 0$. Similarly $(1-f)Rf = 0$. This proves that every idempotent of R is central. Now, since R is semiperfect,

$$R = \sum_{i=1}^{n} \oplus\, e_iR$$

where $\{e_i,\ 1 \leqslant i \leqslant n\}$ is a set of orthogonal indecomposable idempotents. Then R is a direct sum of the local rings e_iRe_i, $1 \leqslant i \leqslant n$. By Lemma 1.4, each e_iRe_i is a right q-ring. Thus each e_iRe_i is a duo ring by Corollary 1.3. This proves the result in case $l(J) = 0$.

Case (ii)—$l(J) \neq 0$. Let $l(J)$ be large in eR for some idempotent e of R. We claim that $Ae \neq 0$ for every mini ideal A of R. By Proposition 2.5, $A = gl(J)$ for some indecomposable idempotent g of R such that $eg = g$. Now

$$\begin{aligned} gJg \cap l(J) \subset gRg \cap l(J) &= gl(J)g \\ &= Ag = Aeg \end{aligned}$$

Hence $Ae = 0$ implies that $gJg \cap l(J) = 0$. But since gJg is a right ideal of R, by Lemma 1.6, and $l(J) \subset' eR$, then $gJg = 0$. Since gJg is the Jacobson radical of the local ring gRg, it follows that gRg is a division ring, a contradiction.

Hence $Ae \neq 0$, as desired.

Now, since A is a mini ideal,

$$Ae \neq 0 \text{ implies } Ae = A.$$

Therefore $l(J)e = l(J)$. Also $l(J) \subset eR$ implies that $el(J) = l(J)$. Now, in view of Lemma 2.6, $eR(1-e) \subset l(J)$ and $(1-e)Re \subset l(J)$.
Hence

$$eR(1-e) = 0 = (1-e)Re$$

proving that e is central. Set $S = eR$ and $T = (1-e)R$. Then R is a direct sum of the two rings S and T. By Lemma 1.4, it follows that both S and T are semilocal right q-rings bound to their radicals. Also, it is clear that the right socle of S is large in S and T has zero right socle. Then, by case (i), T is a finite direct sum of local right self-injective duo rings. On the other hand, S is a right PF-ring. Since gSg is not a division ring for every idempotent g of S, every indecomposable idempotent of S is central, by Proposition 1.7. Hence S is a finite direct sum of local right self-injective duo rings. This completes the proof.

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