INTEGRALS OF SCHLÄFLI'S FORM INVOLVING GENERALIZED BESSEL COEFFICIENTS

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In this note Schläfli's integral for the Bessel coefficients of one argument has been extended to the generalized Bessel coefficients.

1. Introduction and Definitions

The generalized Bessel coefficients $J_m^{(1)}\{x_n\}$, $J_m^{(2)}\{x_n\}$ and the generalized modified Bessel coefficients $I_m^{(1)}\{x_n\}$, $I_m^{(2)}\{x_n\}$ are successively defined by the equalities (Gupta 1964, 1966)

$$e^{\frac{1}{2}x_1\left(u-\frac{1}{u}\right)+\ldots+\frac{1}{2}x_n\left(u^n-\frac{1}{u^n}\right)} = \sum_{m=-\infty}^{\infty} u^m J_m^{(1)}\{x_n\} \qquad (1)$$

$$e^{\frac{1}{2}x_1\left(u-\frac{1}{u}\right)+\ldots+\frac{1}{2}x_n\left(u^n+\frac{(-1)^n}{u^n}\right)} = \sum_{m=-\infty}^{\infty} u^m J_m^{(2)}\{x_n\} \qquad (2)$$

$$e^{\frac{1}{2}x_1\left(u+\frac{1}{u}\right)+\ldots+\frac{1}{2}x_n\left(u^n+\frac{1}{u^n}\right)} = \sum_{m=-\infty}^{\infty} u^m I_m^{(1)}\{x_n\} \qquad (3)$$

$$e^{\frac{1}{2}x_1\left(u+\frac{1}{u}\right)+\ldots+\frac{1}{2}x_n\left(u^n+\frac{(-1)^{n+1}}{u^n}\right)} = \sum_{m=-\infty}^{\infty} u^m I_m^{(2)}\{x_n\} \qquad (4)$$

valid for all finite values of x's and u provided $u \neq 0$.

Integral representations of these coefficients are given by

$$J_m^{(1)}\{x_n\} = \frac{1}{\pi}\int_0^{\pi} \cos\,(m\theta - x_1 \sin\theta - \ldots - x_n \sin n\theta)\, d\theta \qquad (5)$$

$$J_m^{(2)}\{x_n\} = \frac{1}{\pi}\int_0^{\pi} e^{x_2\cos 2\theta + x_4 \cos 4\theta + \cdots} \cos\,(m\theta - x_1\sin\theta - x_3 \sin 3\theta - \ldots)\, d\theta \qquad (6)$$

$$I_m^{(1)}\{x_n\} = \frac{1}{\pi}\int_0^{\pi} e^{x_1\cos\theta + \ldots + x_n \cos n\theta} \cos m\theta\, d\theta \qquad (7)$$

$$I_m^{(2)}\{x_n\} = \frac{1}{\pi}\int_0^{\pi} e^{x_1\cos\theta + x_3 \cos 3\theta + \cdots} \cos\,(m\theta - x_2\sin 2\theta - x_4 \sin 4\theta - \ldots)\, d\theta \qquad (8)$$

where $m = 0, \pm 1, \pm 2, \ldots$.
The coefficients $J_m^{(1)}\{x_n\}$, $J_m^{(2)}\{x_n\}$ are generalizations of $J_m(x_1)$ and $I_m^{(1)}\{x_n\}$, $I_m^{(2)}\{x_n\}$ are the generalizations of $I_m(x_1)$. On comparison it is found that

$$I_m^{(2)}\{x_n\} = e^{-\frac{1}{2}m\pi i} J_m^{(1)}\{x_n e^{\frac{1}{2}n\pi i}\}, \quad I_m^{(1)}\{x_n\} = e^{-\frac{1}{2}m\pi i} J_m^{(2)}\{x_n e^{\frac{1}{2}n\pi i}\}. \quad \ldots \quad (9)$$

Integrals similar to Neumann's form for $J_m^2(x)$ have been given for $(J_m^{(1)}\{x_n\})^2$, $(J_m^{(2)}\{x_n\})^2$, $(I_m^{(1)}\{x_n\})^2$ and $(I_m^{(2)}\{x_n\})^2$ by Gupta (1964). In what follows certain other integrals similar to Schläfli's form (Watson 1944) for $J_m^2(x)$ are given.

2. Integrals

From (5) we have

$$J_m^{(1)}\{x_n\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m\xi - x_1 \sin\xi - \ldots - x_n \sin n\xi)}\, d\xi \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

and so

$$J_m^{(1)}\{x_n\} J_{-m}^{(1)}\{x_n\} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{im(\xi-\eta) - ix_1(\sin\xi + \sin\eta) - \ldots - ix_n(\sin n\xi + \sin n\eta)}\, d\xi\, d\eta.$$

In this, on making the transformation $\xi - \eta = 2\theta$, $\xi + \eta = 2\phi$, it follows that

$$J_m^{(1)}\{x_n\} J_{-m}^{(1)}\{x_n\} = \frac{1}{2\pi^2}\iint e^{2mi\theta - 2ix_1 \sin\phi\cos\theta - \ldots - 2ix_n \sin n\phi \cos n\theta}\, d\phi\, d\theta$$

where the field of integration is the square for which

$$-\pi < \theta - \phi < \pi, \quad -\pi \leqslant \theta + \phi \leqslant \pi.$$

Since the integrand of the last integral is unaffected if both θ and ϕ are increased by π, or if θ is increased by π while ϕ is decreased by π, therefore, the field of integration shall be taken as a rectangle given by

$$0 \leqslant \theta < \pi, \quad -\pi < \phi < \pi.$$

Hence

$$J_m^{(1)}\{x_n\} J_{-m}^{(1)}\{x_n\} = \frac{1}{2\pi^2}\int_0^{\pi}\int_{-\pi}^{\pi} e^{2mi\theta - 2ix_1 \sin\phi\cos\theta - \ldots - 2ix_n \sin n\phi \cos n\theta}\, d\phi\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{2mi\theta} J_0^{(1)}\{2x_n \cos n\theta\}\, d\theta. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

From (1)–(4), on readjustments, we note that

$$J_m^{(1)}\{x_n\} = J_{-m}^{(1)}\{-x_n\} = (-1)^m J_m^{(1)}\{(-1)^n x_n\} \quad \ldots \quad \ldots \quad (12)$$

$$J_m^{(2)}\{x_n\} = J_{-m}^{(2)}\{(-1)^n x_n\} = (-1)^m J_m^{(2)}\{(-1)^n x_n\} \quad \ldots \quad \ldots \quad (13)$$

$$I_m^{(1)}\{x_n\} = I_{-m}^{(1)}\{x_n\} = (-1)^m I_m^{(1)}\{(-1)^n x_n\} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

$$I_m^{(2)}\{x_n\} = I_{-m}^{(2)}\{(-1)^{n+1} x_n\} = (-1)^m I_m^{(2)}\{(-1)^n x_n\}. \quad \ldots \quad (15)$$

Thus, in view of (12), (11) gives

$$J_m^{(1)}\{x_n\}J_{-m}^{(1)}\{x_n\} = \frac{1}{\pi}\int_0^\pi J_0^{(1)}\{2x_n \cos n\theta\} \cos 2m\theta \, d\theta \quad \ldots \quad \ldots \quad (16)$$

$$= \frac{(-1)^m}{\pi}\int_0^\pi J_0^{(1)}\left\{2x_n \cos n\left(\frac{\pi}{2}-\theta\right)\right\} \cos 2m\theta \, d\theta. \quad \ldots \quad (17)$$

In a similar manner the following results are obtained

$$\left(J_m^{(2)}\{x_n\}\right)^2 = \frac{(-1)^m}{\pi}\int_0^\pi J_0^{(2)}\{2x_n \cos n\theta\} \cos 2m\theta \, d\theta \quad \ldots \quad \ldots \quad (18)$$

$$= \frac{1}{\pi}\int_0^\pi J_0^{(2)}\left\{2x_n \cos n\left(\frac{\pi}{2}-\theta\right)\right\} \cos 2m\theta \, d\theta \quad \ldots \quad \ldots \quad (19)$$

$$I_m^{(2)}\{x_n\}I_{-m}^{(2)}\{x_n\} = \frac{1}{\pi}\int_0^\pi I_0^{(2)}\{2x_n \cos n\theta\} \cos 2m\theta \, d\theta \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$= \frac{(-1)^m}{\pi}\int_0^\pi I_0^{(2)}\left\{2x_n \cos n\left(\frac{\pi}{2}-\theta\right)\right\} \cos 2m\theta \, d\theta \quad \ldots \quad (21)$$

$$\left(I_m^{(1)}\{x_n\}\right)^2 = \frac{(-1)^m}{\pi}\int_0^\pi I_0^{(1)}\{2x_n \cos n\theta\} \cos 2m\theta \, d\theta \quad \ldots \quad \ldots \quad (22)$$

$$= \frac{1}{\pi}\int_0^\pi I_0^{(1)}\left\{2x_n \cos n\left(\frac{\pi}{2}-\theta\right)\right\} \cos 2m\theta \, d\theta. \quad \ldots \quad (23)$$

Results (11) and (16)–(23) give the required integrals of the Schläfli's form.

References

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