FOURIER SERIES FOR H-FUNCTION

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In this paper an attempt has been made to derive a Fourier series expansion of the H-function defined by Fox. The series is analogous to that of other special functions such as the MacRobert E-function and the Meijer G-function given by MacRobert, Narayan, Parihar and Anandani. Certain expansions given by them and a few others may be arrived at either by using the properties of the H-function or by particularizing the parameters involved in the main result of this paper. In the end two integrals have been evaluated by making use of the results established in earlier sections.

§ 1. Recently the Fourier series expansions for MacRobert's E-function, Meijer's G-function and Fox's H-function have been given by MacRobert (1961), Narayan (1966), Parihar (1969) and Anandani (1969). In the present note the authors have given another Fourier series expansion for the H-function. This result being more general in character gives rise to the results mentioned above on particularizing the parameters involved therein.

§ 2. The following Fourier series expansion is proposed to be established:

$$\sum_{r=0}^{\infty} \frac{(k+r)\,!}{k\,!\,r\,!} H^{m+2,\,n+1}_{p+3,\,q+3}\left[z \middle| \begin{matrix} (1-r,\,h),\,\{(a_p,\,e_p)\},\,(1,\,h),\,(k+r+2,\,h) \\ \left(1+\frac{k}{2},\,h\right),\,(\frac{3}{2}+\frac{1}{2}k,\,h),\,\{(b_q,\,f_q)\},\,(1,\,h) \end{matrix} \right] \sin\,(k+2r+1)\theta$$

$$= \frac{\sqrt{\pi}}{2} \sum_{d=0}^{k} \frac{\sin\,\theta(\cos\,\theta-1)^d}{d\,!\,(k-d)\,!} H^{m+2,\,n}_{p+2,\,q+2}\left[\frac{z}{\sin^{2h}\theta} \middle| \begin{matrix} \{(a_p,\,e_p)\},\,(1,\,h),\,(\frac{3}{2}+d,\,h) \\ \left(1+\frac{k+d}{2},\,h\right),\,\left(\frac{3}{2}+\frac{k+d}{2},\,h\right),\,\{(b_q,\,f_q)\} \end{matrix} \right], \quad \text{.. (2.1)}$$

where $0 \leqslant \theta \leqslant \pi$, $|\arg z| < (m+n-\frac{1}{2}p-q)\pi$, $p+q < 2(m+n)$ and the H-function is as defined by Fox (1961).

We give below some results, which will be required in the proof of the main result.

$$(\sin\,\theta)^{1-2s} P^{1-s}_{n}(\cos\,\theta) = \sum_{r=0}^{\infty} \frac{2^{2s}(n+r)\,!\,\Gamma(n+2-2s)\,\Gamma(r+s)}{\Gamma(s)\,\Gamma(1-s)r\,!\,\Gamma(n+r+2-s)} \sin\,(n+2r+1)\theta, \qquad (2.2)$$

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Note: The symbols used in this paper have their usual meanings as assigned by previous writers.

where $s < 1$, $0 \leqslant \theta \leqslant \pi$ and $P_n(\cos\theta)$ is given by

$$\sum_{n=0}^{\infty} P_n^{\lambda}(\cos\theta)\,.\,r^n = (1-2r\cos\theta+r^2)^{-\lambda}, \qquad \text{(2.3)}$$

$$P_n^{\lambda}(z) = \sum_{m=0}^{n} \frac{(2\lambda)_{m+n}\left(\dfrac{z-1}{2}\right)^m}{m!\,(n-m)!\,(\lambda+\frac{1}{2})_m}, \qquad \text{(2.4)}$$

the Legendre duplication formula

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}), \qquad \text{(2.5)}$$

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}\{(a_p, e_p)\}\\ \{(b_q, f_q)\}\end{matrix}\right] = z^{-r} H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}\{(a_p+e_p r, e_p)\}\\ \{(b_q+f_q r, f_q)\}\end{matrix}\right], \qquad \text{(2.6)}$$

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}\{(a_p, 1)\}\\ \{(b_q, 1)\}\end{matrix}\right] = G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}(a_p)\\ (b_q)\end{matrix}\right) \qquad \text{(2.7)}$$

and

$$H_{q+1,p}^{p,1}\left[z\,\middle|\,\begin{matrix}(1,1), \{(b_q, 1)\}\\ \{(a_p, 1)\}\end{matrix}\right] = E[p, (a_p);\, q, (b_q);\, z]. \qquad \text{(2.8)}$$

Proof of (2.1): Using the contour integral form of the H-function on the L.H.S. of (2.1), we get

$$\text{L.H.S.} = \sum_{r=0}^{\infty} \frac{(k+r)!}{k!\,r!} \sin(k+2r+1)\theta\, \frac{1}{2\pi i} \int_L \frac{\Gamma(1+\frac{1}{2}k-hs)\,\Gamma(\frac{3}{2}+\frac{1}{2}k-hs)\,\Gamma(r+hs)}{\Gamma(hs)\,\Gamma(1-hs)\,\Gamma(k+r+2-hs)}$$
$$\times \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)}\, z^s\, ds, \qquad \text{(2.9)}$$

where L, the path of integration, runs from $c-i\infty$ to $c+i\infty$. The conditions

$$0 < hc < \tfrac{1}{2}k+1, \quad \operatorname{Re}\left(\frac{b_j}{f_j}\right) > c \quad (j = 1, 2, \ldots, m), \text{ and}$$

$$\operatorname{Re}\left(\frac{a_i-1}{e_i}\right) < c \quad (i = 1, 2, \ldots, n)$$

ensure that all the poles of $\Gamma(1+\frac{1}{2}k-hs)$, $\Gamma(\frac{3}{2}+\frac{1}{2}k-hs)$ and $\Gamma(b_j-f_j s)$ $(j = 1, 2, \ldots, m)$ lie to the right and those of $\Gamma(1-a_i+e_i s)$ $(i = 1, 2, \ldots, n)$ lie to the left of the contour. The integral converges if $p+q < 2(m+n)$ and $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$. Now, changing the order of integration and summation in (2.9), which is justifiable, and putting

$$\frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)} = A$$

we find that

$$\text{L.H.S.} = \frac{1}{2\pi i}\int_L A \sum_{r=0}^{\infty} \frac{(k+r)!\,\Gamma(1+\frac{1}{2}k-hs)\,\Gamma(\frac{3}{2}+\frac{1}{2}k-hs)\,\Gamma(r+hs)}{k!\,r!\,\Gamma(hs)\,\Gamma(1-hs)\,\Gamma(k+r+2-hs)} \times \sin(k+2r+1)\theta\,.\,z^s\,ds$$

$$= \frac{1}{2\pi i}\int_L A \frac{\sqrt{\pi}}{2^{k+1}}(\sin\theta)^{1-2hs} P_k^{1\ \ hs}(\cos\theta) z^s\,ds$$

$$= \frac{1}{2\pi i}\int_L A \frac{\sqrt{\pi}}{2^{k+1}} \sum_{d=0}^{k} \frac{(\sin\theta)^{1-2hs}(2-2hs)_{d+k}}{d!\,(k-d)!\,(\frac{3}{2}-hs)_d}\left(\frac{\cos\theta-1}{2}\right)^d z^s\,ds$$

on using (2.2), (2.5) and (2.4) one after the other. A little more simplification yields

$$\sum_{d=0}^{k} \frac{\sqrt{\pi}}{2}\cdot\frac{\sin\theta\,(\cos\theta-1)^d}{d!\,(k-d)!}\,\frac{1}{2\pi i}\int_L \frac{\Gamma\left(1+\frac{k+d}{2}-hs\right)\Gamma\left(\frac{3}{2}+\frac{k+d}{2}-hs\right)}{\Gamma(1-hs)\,\Gamma(\frac{3}{2}+d-hs)} \times A\left(\frac{z}{\sin^2\theta}\right)^{hs} ds$$

which, when thrown into the form of the H-function, gives the R.H.S. Hence the result.

§ 3. In this section we shall derive certain Fourier series for other special functions either by applying the property of the H-function or by specializing the parameters.

(i) On using the transformation (2.6), the following Fourier series for the H-function is arrived at:

$$\sum_{r=0}^{\infty} \frac{(k+r)!}{k!\,r!} H_{p+3,\,q+3}^{m+2,\,n+1}\left[z \,\middle|\, \begin{matrix}(1-r+hr,\,h),\,\{(a_p+re_p,\,e_p)\},\,(1+hr,\,h),\,(k+2+r+hr,\,h)\\ (1+\frac{1}{2}k+hr,\,h),\,(\frac{3}{2}+\frac{1}{2}k+hr,\,h),\,\{(b_q+rf_q,\,f_q)\},\,(1+hr,\,h)\end{matrix}\right] \times \sin(k+2r+1)\theta\,.\,z^{-r}$$

$$= \frac{\sqrt{\pi}}{2}\sum_{d=0}^{k} \frac{\sin\theta\,(\cos\theta-1)^d}{d!\,(k-d)!}\, H_{p+2,\,q+2}^{m+2,\,n} \times\left[\frac{z}{\sin^{2h}\theta}\,\middle|\, \begin{matrix}\{(a_p,\,e_p)\},\,(1,\,h),\,(\frac{3}{2}+d,\,h)\\ \left(1+\frac{k+d}{2},\,h\right),\,\left(\frac{3}{2}+\frac{k+d}{2},\,h\right),\,\{(b_q,\,f_q)\}\end{matrix}\right]. \quad \cdots (3.1)$$

(ii) Taking $k = 0$ in (2.1), we obtain

$$\sum_{r=0}^{\infty} H^{m+1,\,n+1}_{p+2,\,q+2}\left[z\,\middle|\,\begin{matrix}(1-r,\,h),\ \{(a_p,\,e_p)\},\ (r+2,\,h)\\ (\tfrac{3}{2},\,h),\ \{(b_q,\,f_q)\},\ (1,\,h)\end{matrix}\right]\sin(2r+1)\theta$$

$$=\frac{\sqrt{\pi}}{2}\sin\theta\, H^{m,\,n}_{p,\,q}\left[\frac{z}{\sin^{2h}\theta}\,\middle|\,\begin{matrix}\{(a_p,\,e_p)\}\\ \{(b_q,\,f_q)\}\end{matrix}\right], \qquad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

which is a known result due to Anandani (1969).

(iii) On putting $h = 1 = (e_p) = (f_q)$ in (2.1), we get the following known result (Narayan 1966) for the Meijer G-function:

$$\sum_{r=0}^{\infty}\frac{(k+r)!}{k!\,r!}\,G^{m+2,\,n+1}_{p+3,\,q+3}\left(z\,\middle|\,\begin{matrix}1-r,\ (a_p),\ k+r+2\\ 1+\tfrac{1}{2}k,\ \tfrac{3}{2}+\tfrac{1}{2}k,\ (b_q),\ 1\end{matrix}\right)\sin(k+2r+1)\theta$$

$$=\frac{\sqrt{\pi}}{2}\sum_{d=0}^{k}\frac{\sin\theta\,(\cos\theta-1)^d}{d!\,(k-d)!}\,G^{m+2,\,n}_{p+2,\,q+2}\left(\frac{z}{\sin^2\theta}\,\middle|\,\begin{matrix}(a_p),\ 1,\ \tfrac{3}{2}+d\\ 1+\dfrac{k+d}{2},\ \dfrac{3}{2}+\dfrac{k+d}{2},\ (b_q)\end{matrix}\right). \qquad \ldots \quad (3.3)$$

In addition, if we take $k = 0$ in (3.3), we obtain a Fourier series for the Meijer G-function given by Narayan (1966).

(iv) On making use of (2.8), after particularizing the parameters suitably, we obtain the following result (Parihar 1969) for the MacRobert E-function:

$$\sum_{r=0}^{\infty}\frac{(k+r)!}{k!\,r!}E[1+r+\tfrac{1}{2}k,\ \tfrac{3}{2}+r+\tfrac{1}{2}k,\ (a_p)+r;\ 1+r,\ k+2r+2,\ (b_q)+r : z]$$

$$\times\sin(k+2r+1)\theta$$

$$=\frac{\sqrt{\pi}}{2}\sum_{d=0}^{k}\frac{z\sin\theta\,(\cos\theta-1)^d}{d!\,(k-d)!}E\left[1+\frac{k+d}{2},\ \frac{3}{2}+\frac{k+d}{2},\ (a_p);\ 1,\ \tfrac{3}{2}+d,\ (b_q) : \frac{z}{\sin^2\theta}\right]. \qquad \ldots \quad (3.4)$$

In particular, when $k = 0$, (3.4) reduces to a known Fourier series for the E-function due to MacRobert (1961).

§ 4. *Deductions*—From (2.1) and (3.1), we easily deduce the following integrals:

$$\text{(i)}\quad \int_0^{\pi}\sum_{d=0}^{k} H^{m+2,\,n}_{p+2,\,q+3}\left[\frac{z}{\sin^{2h}\theta}\,\middle|\,\begin{matrix}\{(a_p,\,e_p)\},\ (1,\,h),\ (\tfrac{3}{2}+d,\,h)\\ \left(1+\dfrac{k+d}{2},\,h\right),\ \left(\dfrac{3}{2}+\dfrac{k+d}{2},\,h\right),\ \{(b_q,\,f_q)\}\end{matrix}\right]$$

$$\times\frac{\sin\theta\,\sin(k+2r+1)\theta\,(\cos\theta-1)^d}{d!\,(k-d)!}$$

$$= \sum_{r=0}^{\infty} \frac{\sqrt{\pi}(k+r)!}{k!\,r!} H^{m+2,\,n+1}_{p+3,\,q+3}\left[z \left| \begin{matrix} (1-r,\,h),\,\{(a_p,\,e_p)\},\,(1,\,h),\,(k+r+2,\,h) \\ \left(1+\frac{k}{2},\,h\right),\,(\frac{3}{2}+\frac{1}{2}k,\,h),\,\{(b_q,\,f_q)\},\,(1,\,h) \end{matrix} \right. \right] \quad (4.1)$$

and (ii)

$$\int_0^{\pi} \sum_{d=0}^{k} H^{m+2,\,n}_{p+2,\,q+2}\left[\frac{z}{\sin^{2h}\theta} \left| \begin{matrix} \{(a_p,\,e_p)\},\,(1,\,h),\,(\frac{3}{2}+d,\,h) \\ \left(1+\frac{k+d}{2},\,h\right),\,\left(\frac{3}{2}+\frac{k+d}{2},\,h\right),\,\{(b_q,\,f_q)\} \end{matrix} \right. \right]$$

$$\times \frac{\sin\theta\,.\,\sin(k+2r+1)\theta\,.\,(\cos\theta-1)^d}{d!\,(k-d)!}$$

$$= \sum_{r=0}^{\infty} \frac{(k+r)!}{k!\,r!} z^{-r} H^{m+2,\,n+1}_{p+3,\,q+3}\left[z \left| \begin{matrix} (1-r+hr,\,h),\,\{(a_p+re_p,\,e_p)\},\,(1+hr,\,h), \\ (k+2+r+hr,\,h) \\ (1+\frac{1}{2}k+hr,\,h),\,(\frac{3}{2}+\frac{1}{2}k+hr,\,h),\,\{(b_q+rf_q,\,f_q)\}, \\ (1+hr,\,h) \end{matrix} \right. \right], \quad (4.2)$$

where $p+q < 2(m+n)$ and $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$.

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