ON ALMOST PRODUCT SPACES

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In this paper we have obtained some results in the almost product and almost decomposable spaces by making use of a vector valued bilinear function.

1. INTRODUCTION

Let us consider an n-dimensional manifold M_n of differentiability class C^{r+1} endowed with a tensor F or f of type (0, 2) or (1, 1) respectively such that if $X, Y, Z\ldots$ denote the covariant vectors, then

$$\overline{X} = f(X) \quad \ldots \quad (1.1)$$

and

$$g(\overline{X}, Y) = F(X, Y) \quad \ldots \quad (1.2)$$

g being the non-singular metric tensor of M_n. From (1.1), we have

$$\overline{\overline{X}} = X. \quad \ldots \quad (1.3)$$

A space for which (1.3) and

$$g(\overline{X}, \overline{Y}) = g(X, Y) \quad \ldots \quad (1.4)$$

hold, is called almost product space and F or f the almost product structure. In this space, we have

$$F(X, Y) = F(Y, X) \quad \ldots \quad (1.5a)$$

and

$$F(\overline{X}, \overline{Y}) = F(X, Y). \quad \ldots \quad (1.5b)$$

Suppose D_X is a Riemannian connexion in this space, then

$$D_X Y - D_Y X = [X, Y] \quad \ldots \quad (1.6)$$

and

$$(D_X g)(Y, Z) = 0. \quad \ldots \quad (1.7)$$

An almost product space in which

$$(D_X f)(Y) = 0 \quad \ldots \quad (1.8)$$

is satisfied is called an almost product and almost decomposable space.

2. $M(X, Y)$ TENSOR

Let us consider a vector valued bilinear function $M(X, Y)$ given by (Mishra 1969)

$$M(X, Y) = D_{\overline{X}}\overline{Y} + D_X Y - \overline{D_{\overline{X}} Y} - \overline{D_X \overline{Y}}. \quad \ldots \quad (2.1)$$

Theorem (2.1)—If we assume

$$A(X, Y) = D_{\bar{X}}\bar{Y} - \overline{D_{\bar{X}}Y} \quad \ldots \quad (2.2a)$$

then

$$A(\bar{X}, \bar{Y}) = D_X Y - \overline{D_X \bar{Y}} \quad \ldots \quad (2.2b)$$

$$\overline{A(X, Y)} = -A(X, \bar{Y}) \quad \ldots \quad (2.2c)$$

and

$$\overline{A(\bar{X}, \bar{Y})} = -A(\bar{X}, Y). \quad \ldots \quad (2.2d)$$

Consequently

$$M(X, Y) = A(X, Y) + A(\bar{X}, \bar{Y}). \quad \ldots \quad (2.3)$$

Proof: Replacing X and Y in (2.2a) and (2.2c) by $\bar{X}$ and $\bar{Y}$ we get (2.2b) and (2.2d) respectively. Barring Y in (2.2a) and comparing it with $\overline{A(X, Y)}$, we have (2.2c). From (2.1), (2.2a) and (2.2b) we have (2.3).

Theorem (2.2)—We have

$$M(X, Y) = M(\bar{X}, \bar{Y}) \quad \ldots \quad (2.4a)$$

and

$$M(X, Y) = -\overline{M(\bar{X}, Y)} = -\overline{M(X, \bar{Y})}. \quad \ldots \quad (2.4b)$$

Proof: Replacing X, Y in (2.3) by $\bar{X}$, $\bar{Y}$ and using it, we get (2.4a). Barring the result obtained by replacing Y in (2.3) by $\bar{Y}$ and using (2.2), we have (2.4b).

Corollary 2.1—We have

$$M(\bar{X}, Y) = M(X, \bar{Y}) = -\overline{M(\bar{X}, \bar{Y})} = -\overline{M(X, Y)}. \quad \ldots \quad (2.5)$$

Proof: Replacing X and Y in (2.4) by $\bar{X}$ and $\bar{Y}$, we have (2.5).

Let us define

$$A(X, Y, Z) = g(A(X, Y), Z) \quad \ldots \quad (2.6)$$

and

$$M(X, Y, Z) = g(M(X, Y), Z). \quad \ldots \quad (2.7a)$$

Consequently

$$M(X, Y, Z) = A(X, Y, Z) + A(\bar{X}, \bar{Y}, Z). \quad \ldots \quad (2.7b)$$

Theorem (2.3)—We have

$$M(X, Y, Z) = M(\bar{X}, \bar{Y}, Z) \quad \ldots \quad (2.8a)$$

$$M(X, Y, Z) = -M(X, \bar{Y}, \bar{Z}) \quad \ldots \quad (2.8b)$$

and

$$M(X, Y, Z) = -M(\bar{X}, Y, \bar{Z}). \quad \ldots \quad (2.8c)$$

Proof: Using (2.4a) in (2.7a) we obtain (2.8a). With the help of (1.2), (1.4) and (2.4b), (2.7a) yields (2.8b) and (2.8c).

Corollary 2.2—We have

$$M(\bar{X}, Y, Z) = M(X, \bar{Y}, Z) = -M(X, Y, \bar{Z}). \quad \ldots \quad (2.9)$$

Proof: Replacing X in (2.8a) by $\bar{X}$ and Y in (2.8b) by $\bar{Y}$, we get (2.9).

3. Almost Product and Almost Decomposable Space

Theorem (3.1)—The necessary and sufficient condition that an almost product space to be an almost product and almost decomposable space is

$$A(X, Y) = 0. \qquad (3.1)$$

Proof: For an almost product space

$$(D_X f)(Y) = 0.$$

With the help of this, we have

$$D_X \overline{Y} = \overline{D_X Y} \qquad (3.2)$$

which is equivalent to (by barring X in (3.2))

$$D_{\overline{X}} \overline{Y} = \overline{D_{\overline{X}} Y}. \qquad (3.3)$$

By virtue of (2.2), (3.3) gives (3.1).

Conversely, if (3.1) is satisfied, then (3.3) holds which is same as (3.2). Putting this value in

$$D_X \overline{Y} = (D_X f)(Y) + \overline{D_X Y}$$

we have

$$(D_X f)(Y) = 0.$$

Hence Theorem 3.1 is proved.

Theorem (3.2)—The necessary and sufficient condition that an almost product space to be an almost product and almost decomposable is

$$M(X, Y) = 0. \qquad (3.4)$$

Proof: From (2.2) and (3.1) we have (3.4).

4. Nijenhuis Tensor

The Nijenhuis tensor $N(X, Y)$ is defined by

$$N(X, Y) = [\overline{X}, \overline{Y}] + [X, Y] - \overline{[\overline{X}, Y]} - \overline{[X, \overline{Y}]}. \qquad (4.1)$$

By virtue of (2.1) it is equivalent to

$$N(X, Y) = M(X, Y) - M(Y, X). \qquad (4.2)$$

Using (2.2) we have

$$N(X, Y) = A(X, Y) - A(Y, X) + A(\overline{X}, \overline{Y}) - A(\overline{Y}, \overline{X}). \qquad (4.3)$$

Let us define

$$N(X, Y, Z) = g(N(X, Y), Z). \qquad (4.4)$$

It can be shown with the help of (2.4) and (2.8) that

$$N(X, Y) = N(\overline{X}, \overline{Y}) = -\overline{N(X, \overline{Y})} \qquad (4.5a)$$

$$N(\overline{X}, Y) = N(X, \overline{Y}) = \overline{N(\overline{X}, \overline{Y})} = \overline{N(X, Y)} \qquad (4.5b)$$

$$N(X, Y, Z) = N(\overline{X}, \overline{Y}, Z) = -N(X, \overline{Y}, \overline{Z}) = -N(\overline{X}, Y, \overline{Z}) \qquad (4.5c)$$

and

$$N(\overline{X}, Y, Z) = N(X, \overline{Y}, Z) = -N(X, Y, \overline{Z}). \qquad (4.5d)$$

Theorem (*4.1*)—The necessary and sufficient condition that an almost product space to be an almost product and almost decomposable is

$$N(X, Y) = 0.$$

PROOF: From (3.4) and (4.2) the statement follows:
This theorem has been proved by Yano (1965) by another method.

REFERENCES

Mishra, R. S. (1969). On almost product and almost decomposable manifolds. *Tensor* (*in press*).

Yano, K. (1965). Differential Geometry on Complex and Almost Complex Spaces. Pergamon Press, New York.