SLOW UNSTEADY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID THROUGH A CIRCULAR TUBE WITH AXIAL ROUGHNESS

by P. D. VERMA, *Department of Mathematics, University of Rajasthan, Jaipur*

and

Y. N. GAUR, *Department of Mathematics, M.R. Engineering College, Jaipur*

(Communicated by P. L. Bhatnagar, F.N.I.)

(*Received 6 October 1969*)

The problem of slow unsteady flow of a viscous incompressible fluid through a circular tube with axial roughness has been investigated under the assumption that the roughness is small. The integral transform technique has been used to obtain the axial and radial velocity components and the pressure. A particular case of sinusoidal roughness has been discussed numerically.

1. INTRODUCTION

The exact analytical solutions of linearized Navier-Stokes' equations, introducing the assumption of axially parallel flow of a viscous incompressible fluid through a tube of circular cross-section, under the influence of periodic pressure gradient, have been obtained by Sexl (1930) and Uchida (1956); while those for coaxial circular cylinders by Verma (1960). The slow viscous flow between rotating concentric infinite cylinders with axial roughness has been discussed by Citron (1962) and the problems of flow of non-Newtonian fluids and heat transfer between wavy walls and wavy cylinders have been extensively studied by Bhatnagar and Mohan Rao (1965), Bhatnagar and Mathur (1967), and Mathur (1967).

In this paper the solution of slow unsteady flow of a viscous incompressible fluid through a circular tube with axial roughness, under the influence of pulsating pressure gradient, has been investigated. The roughness has been taken to be small in comparison to the smooth radius of the tube. The integral transform technique has been extensively made use of for finding the solutions. Expressions for axial velocity, radial velocity and pressure have been obtained in terms of modified Bessel functions. The case of sinusoidal roughness has been studied in particular with numerical investigation.

2. Problem Formulation

The Navier-Stokes' equations of a viscous incompressible fluid neglecting the external forces are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u, \qquad (2.1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v, \qquad (2.2)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w, \qquad (2.3)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \qquad (2.4)$$

where

$$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

As we are considering the slow motion, neglecting the inertial terms in (2.1) to (2.3) and making use of (2.4), we have

$$\nabla^2 p=0, \qquad (2.5)$$

and eqn. (2.3) then gives

$$\frac{\partial}{\partial t}(\nabla^2 w)=\nu\nabla^4 w. \qquad (2.6)$$

In cylindrical polar coordinates (r, θ, z) with the velocity components (u_r, u_θ, w) in the increasing directions of r, θ and z respectively, eqns. (2.1) to (2.4) are reduced to

$$\frac{\partial u_r}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\nabla^2 u_r-\frac{u.r}{r^2}\right], \qquad (2.7)$$

$$\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w, \qquad (2.8)$$

and

$$\frac{1}{r}\frac{\partial(u_r.r)}{\partial r}+\frac{\partial w}{\partial z}=0, \qquad (2.9)$$

where

$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

Introducing the dimensionless quantities

$$U=\frac{au_r}{\nu},\quad W=\frac{aw}{\nu},\quad \lambda=\frac{r}{a},\quad Z=\frac{z}{a},\quad P=\frac{pa^2}{\nu^2\rho} \qquad (2.10)$$

and

$$T=\frac{t\nu}{a^2},$$

where a is the smooth radius of the tube; eqns. (2.7) to (2.9) are transformed to

$$\frac{\partial U}{\partial T} = -\frac{\partial P}{\partial \lambda} + \frac{\partial^2 U}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial U}{\partial \lambda} + \frac{\partial^2 U}{\partial Z^2} - \frac{U}{\lambda^2}, \quad \ldots \quad \ldots \quad (2.11)$$

$$\frac{\partial W}{\partial T} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial W}{\partial \lambda} + \frac{\partial^2 W}{\partial Z^2}, \quad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

and

$$\frac{1}{\lambda}\frac{\partial (U\lambda)}{\partial \lambda} + \frac{\partial W}{\partial Z} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.13)$$

In cylindrical polar coordinates (2.6) is transformed to

$$\left(\frac{\partial^2}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial}{\partial \lambda} + \frac{\partial^2}{\partial Z^2}\right)^2 W = \frac{\partial}{\partial T}\left(\frac{\partial^2}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial}{\partial \lambda} + \frac{\partial^2}{\partial Z^2}\right) W. \quad \ldots \quad \ldots \quad (2.14)$$

The boundary conditions are

$$U = W = 0 \text{ at } \lambda = 1 + \epsilon N(Z), \quad T > 0, \quad Z > 0,$$

where $\epsilon \ll 1$ is the roughness parameter.

Let

$$P(\lambda, Z, T) = P_0(Z, T) + P'(\lambda, Z, T), \quad \ldots \quad \ldots \quad (2.15)$$

$$U(\lambda, Z, T) = U'(\lambda, Z, T), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.16)$$

and

$$W(\lambda, Z, T) = W_0(\lambda, T) + W'(\lambda, Z, T), \quad \ldots \quad \ldots \quad (2.17)$$

where the primed quantities are the variations caused by the roughness and P_0 and W_0 are the known quantities for the cylinder without roughness, given by

$$\frac{\partial P_0}{\partial \lambda} = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.18)$$

and

$$\frac{\partial W_0}{\partial T} = -\frac{\partial P_0}{\partial Z} + \frac{\partial^2 W_0}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial W_0}{\partial \lambda}. \quad \ldots \quad \ldots \quad \ldots \quad (2.19)$$

We shall assume that the pressure gradient caused by the motion of the piston is harmonic and is given by

$$-\frac{\partial P_0}{\partial Z} = K \cos nT = \text{Real part of } K \,.\, e^{inT}, \quad \ldots \quad \ldots \quad (2.20)$$

where K denotes a constant and

$$W_0 = f(\lambda) \cos nT = \text{Real part of } f(\lambda) \,.\, e^{inT}. \quad \ldots \quad \ldots \quad (2.21)$$

In light of (2.20) and (2.21), (2.19) becomes

$$f''(\lambda) + \frac{1}{\lambda} f'(\lambda) - in\, f(\lambda) = -K, \quad \ldots \quad \ldots \quad \ldots \quad (2.22)$$

the solution of which is

$$W_0(\lambda, T) = \text{Real part of } \left[-\frac{iK}{n} e^{inT}\left\{1 - \frac{J_0(\lambda n^{\frac{1}{2}} i^{\frac{3}{2}})}{J_0(n^{\frac{1}{2}} i^{\frac{3}{2}})}\right\}\right] \quad \ldots \quad (2.23)$$

which for very slow oscillations reduces to

$$W_0(\lambda, T) = \frac{K}{4}(1-\lambda^2) \cos nT = \text{Re}\left[\frac{K}{4}(1-\lambda^2)e^{inT}\right]. \quad .. \ (2.24)$$

From (2.11) to (2.13) and (2.15) to (2.17), we have

$$\frac{\partial U'}{\partial T} = -\frac{\partial P'}{\partial \lambda}+\frac{\partial^2 U'}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial U'}{\partial \lambda}+\frac{\partial^2 U'}{\partial Z^2}-\frac{U'}{\lambda^2}, \quad .. \ (2.25)$$

$$\frac{\partial W'}{\partial T} = -\frac{\partial P'}{\partial Z}+\frac{\partial^2 W'}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial W'}{\partial \lambda}+\frac{\partial^2 W'}{\partial Z^2}, \quad .. \ (2.26)$$

and

$$\frac{1}{\lambda}\frac{\partial (U'\lambda)}{\partial \lambda}+\frac{\partial W'}{\partial Z} = 0. \quad .. \ (2.27)$$

Differentiating (2.27) with respect to λ and then using (2.25) we get

$$\frac{\partial U'}{\partial T} = -\frac{\partial P'}{\partial \lambda}+\frac{\partial^2 U'}{\partial Z^2}-\frac{\partial^2 W'}{\partial \lambda\, \partial Z}. \quad .. \ (2.28)$$

Eqn. (2.14) is now reduced to

$$\left(\frac{\partial^2}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial}{\partial \lambda}+\frac{\partial^2}{\partial Z^2}\right)^2 W' = \frac{\partial}{\partial T}\left(\frac{\partial^2}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial}{\partial \lambda}+\frac{\partial^2}{\partial Z^2}\right)W' \quad .. \ (2.29)$$

under the boundary conditions

$$\left.\begin{aligned} &U' = 0, \quad W' = -W_0 \ \text{at} \ \lambda = 1+\epsilon N(Z), \quad T > 0, \quad Z > 0 \\ \text{and} \quad &W' = 0 \ \text{when} \ 0 < \lambda < 1+\epsilon N(Z), \quad T < 0, \quad Z = 0 \end{aligned}\right\} \quad .. \ (2.30)$$

3. Method of Solution

Following (2.21), we assume

$$W'(\lambda, Z, T) = W'(\lambda, Z) \cos nT = \text{Re}\,[W'(\lambda, Z)\, e^{inT}]. \quad .. \ (3.1)$$

Eqn. (2.29) is, therefore, reduced to

$$\left(\frac{\partial^2}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial}{\partial \lambda}+\frac{\partial^2}{\partial Z^2}\right)^2 W' = in\left(\frac{\partial^2 W'}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial W'}{\partial \lambda}+\frac{\partial^2 W'}{\partial Z^2}\right). \quad .. \ (3.2)$$

Let

$$\frac{\partial^2 W'}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial W'}{\partial \lambda}+\frac{\partial^2 W'}{\partial Z^2} = f(\lambda, Z). \quad .. \ (3.3)$$

From (3.2) and (3.3) we have

$$\frac{\partial^2 f}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial f}{\partial \lambda}+\frac{\partial^2 f}{\partial Z^2} = in\, f. \quad .. \ (3.4)$$

Let

$$F(\lambda, \xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\lambda, Z) \sin(\xi Z)\, dZ \quad .. \ (3.5)$$

where $F(\lambda, \xi)$ is Fourier sine transform of $f(\lambda, z)$.

Taking Fourier sine transform of (2.4), we have

$$\frac{d^2F}{d\lambda^2}+\frac{1}{\lambda}\frac{dF}{d\lambda}-(\xi^2+in)F=0, \qquad (3.6)$$

the solution of which is

$$F(\lambda,\xi)=B'(\xi)\,I_0(m\lambda), \qquad (3.7)$$

where $m=\sqrt{\xi^2+in}$ and $B'(\xi)$ is the constant of integration.

Taking Fourier sine transform of (3.3), we have

$$\frac{d^2W}{d\lambda^2}+\frac{1}{\lambda}\frac{dW}{d\lambda}-\xi^2W=F(\lambda,\xi), \qquad (3.8)$$

where W is the Fourier sine transform of W'.

From (3.7) and (3.8), we write

$$\frac{d^2W}{d\lambda^2}+\frac{1}{\lambda}\frac{dW}{d\lambda}-\xi^2W=B'(\xi)\,I_0(m\lambda), \qquad (3.9)$$

the complete solution of which is

$$W(\lambda,\xi)=A(\xi)\,I_0(\xi\lambda)+B(\xi)\,I_0(m\lambda), \qquad (3.10)$$

where $A(\xi)$ and $B(\xi)$ are constants of integration.

Taking inverse Fourier sine transform of W, we have

$$W'(\lambda,Z,T)=\mathrm{Re}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty [A(\xi)\,I_0(\xi\lambda)+B(\xi)\,I_0(m\lambda)]\sin(\xi Z)\,e^{inT}\,d\xi\right\}. \qquad (3.11)$$

From (2.27) we have

$$\frac{1}{\lambda}\frac{\partial(U'\lambda)}{\partial\lambda}=-\frac{\partial W'}{\partial Z}$$

$$=\mathrm{Re}\left\{\left(-\sqrt{\frac{2}{\pi}}\right)\int_0^\infty \xi[A(\xi)\,I_0(\xi\lambda)+B(\xi)\,I_0(m\lambda)]\cos(\xi Z)\,e^{inT}\,d\xi\right\}$$

which gives

$$U'(\lambda,Z,T)=\mathrm{Re}\left\{\left(-\sqrt{\frac{2}{\pi}}\right)\int_0^\infty [A(\xi)\,I_1(\xi\lambda)+\frac{\xi}{m}B(\xi)\,I_1(m\lambda)]\cos(\xi Z)\,e^{inT}\,d\xi\right\} \qquad (3.12)$$

using the condition that U' should be finite at the axis.

And from (2.26) and (2.28) using (3.11) and (3.12), we have

$$P'(\lambda,Z,T)=\mathrm{Re}\left\{\sqrt{\frac{2}{\pi}}\,in\int_0^\infty \frac{1}{\xi}A(\xi)\,I_0(\xi\lambda)\cos(\xi Z)\,e^{inT}\,d\xi\right\}+C, \qquad (3.13)$$

where C is the constant of integration.

Using the boundary conditions (2.30) in (3.11) and (3.12), we have

$$-\frac{K}{4}[1-\{1+\epsilon N(Z)\}^2]=\sqrt{\frac{2}{\pi}}\int_0^\infty [A(\xi)\,I_0\{\xi(1+\epsilon N(Z))\}+B(\xi)\,I_0\{m(1+\epsilon N(Z))\}]\sin(\xi Z)\,d\xi \qquad (3.14)$$

and

$$0 = -\sqrt{\frac{2}{\pi}}\int_0^\infty \left[A(\xi)\, I_1\{\xi(1+\epsilon N(Z))\}+\frac{\xi}{m}B(\xi)\, I_1\{m(1+\epsilon N(Z))\}\right] \cos(\xi Z)\, d\xi. \quad (3.15)$$

Let

$$\left.\begin{aligned} A(\xi) &= A_0(\xi)+\epsilon A_1(\xi)+\ldots \\ B(\xi) &= B_0(\xi)+\epsilon B_1(\xi)+\ldots \end{aligned}\right\} \quad (3.16)$$

and

Substituting (3.16) in (3.14) and (3.15) and equating the coefficients of like powers of ϵ, we have, from coefficients of ϵ^0,

$$0 = \sqrt{\frac{2}{\pi}}\int_0^\infty [A_0(\xi)\, I_0(\xi)+B_0(\xi)\, I_0(m)] \sin(\xi Z)\, d\xi$$

and

$$0 = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[A_0(\xi)\, I_1(\xi)+\frac{\xi}{m}B_0(\xi)\, I_1(m)\right] \cos(\xi Z)\, d\xi.$$

Inverting the above two equations by Fourier sine and cosine integral theorems, we have

$$\left.\begin{aligned} A_0(Z)\, I_0(Z)+B_0(Z)\, I_0(m') &= 0 \\ A_0(Z)\, I_1(Z)+\frac{Z}{m'}B_0(Z)\, I_1(m') &= 0 \end{aligned}\right\} \quad (3.17)$$

and

where

$$m' = \sqrt{Z^2+in}.$$

From eqns. (3.17), we have

$$A_0(Z) = B_0(Z) = 0 \quad (3.18)$$

and from coefficients of ϵ, we have

$$\frac{K}{2}N(Z) = \sqrt{\frac{2}{\pi}}\int_0^\infty [A_1(\xi)\, I_0(\xi)+B_1(\xi)\, I_0(m)] \sin(\xi Z)\, d\xi,$$

and

$$0 = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[A_1(\xi)\, I_1(\xi)+\frac{\xi}{m}B_1(\xi)\, I_1(m)\right] \cos(\xi Z)\, d\xi.$$

Inverting the above two equations by Fourier sine and cosine integral theorems, we have

$$\frac{K}{2}\bar{N}(Z) = A_1(Z)\, I_0(Z)+B_1(Z)\, I_0(m') \quad (3.19)$$

and

$$0 = A_1(Z)\, I_1(Z)+\frac{Z}{m'}B_1(Z)\, I_1(m'). \quad (3.20)$$

From (3.19) and (3.20) we get

$$\left.\begin{aligned} A_1(Z) &= -\frac{\frac{K}{2}\overline{N}(Z)\,I_1(m')\,\frac{Z}{m'}}{I_0(m')\,I_1(Z)-\frac{Z}{m'}I_1(m')\,I_0(Z)} \\ \text{and} \qquad B_1(Z) &= \frac{\frac{K}{2}\overline{N}(Z)\,I_1(Z)}{I_0(m')\,I_1(Z)-\frac{Z}{m'}I_1(m')\,I_0(Z)} \end{aligned}\right\} \qquad (3.21)$$

Making use of (3.18) and (3.21) in (3.11), (3.12) and (3.13) and collecting only the real part, the complete expressions for the axial velocity, radial velocity and pressure are

$$\begin{aligned} W(\lambda, Z, T) &= W_0(\lambda, T)+W'(\lambda, Z, T) \\ &= \frac{K}{4}(1-\lambda^2)\cos nT-\frac{K}{2}\epsilon\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\frac{I_2(\xi)\,I_0(\xi\lambda)-\lambda I_1(\xi)\,I_1(\xi\lambda)}{I_1^2(\xi)-I_0(\xi)\,I_2(\xi)}\right] \\ &\qquad \times\overline{N}(\xi)\sin(\xi Z)\cos nT\,d\xi \end{aligned} \qquad (3.22)$$

$$\begin{aligned} U(\lambda, Z, T) &= U'(\lambda, Z, T) \\ &= \frac{K}{2}\epsilon\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\frac{I_0(\xi)\,I_1(\xi\lambda)-\lambda I_1(\xi)\,I_0(\xi\lambda)}{I_1^2(\xi)-I_0(\xi)\,I_2(\xi)}\right]\overline{N}(\xi)\cos(\xi Z)\cos nT\,d\xi \end{aligned} \qquad (3.23)$$

and

$$\begin{aligned} P(\lambda, Z, T) &= P_0(Z, T)+P'(\lambda, Z, T) \\ &= C-KZ\cos nT-\epsilon\sqrt{\frac{2}{\pi}}K\int_0^\infty\frac{I_1(\xi)\,I_0(\xi\lambda)\,\overline{N}(\xi)}{I_1^2(\xi)-I_0(\xi)\,I_2(\xi)}\cos(\xi Z)\cos nT\,d\xi \end{aligned} \qquad (3.24)$$

where C denotes a constant.

4. Particular Case (Sinusoidal Roughness)

For this case we take the roughness function at the walls

$$N(Z) = \sin\frac{Z}{l}, \qquad (4.1)$$

where $2\pi l$ is the wavelength of roughness at the walls.

We may formally write

$$\overline{N}(\xi) = \sqrt{\frac{\pi}{2}}\,\delta\left(\xi-\frac{1}{l}\right) \qquad (4.2)$$

where $\overline{N}(\xi)$ is Fourier sine transform of $N(z)$ and δ is the Dirac delta function.

Substituting $\bar{N}(\xi)$ from (4.2) in (3.22), (3.23) and (3.24) and making use of a property of Dirac delta function (Sneddon 1951), we have

$$W(\lambda, Z, T) = \frac{K}{4}(1-\lambda^2)\cos nT - \frac{K}{2}\epsilon\frac{I_2\left(\frac{1}{l}\right)I_0\left(\frac{\lambda}{l}\right)-\lambda I_1\left(\frac{1}{l}\right)I_1\left(\frac{\lambda}{l}\right)}{I_1^2\left(\frac{1}{l}\right)-I_0\left(\frac{1}{l}\right)I_2\left(\frac{1}{l}\right)}\sin\left(\frac{Z}{l}\right)\cos nT \quad \text{.. (4.3)}$$

$$U(\lambda, Z, T) = \frac{K}{2}\epsilon\frac{I_0\left(\frac{1}{l}\right)I_1\left(\frac{\lambda}{l}\right)-\lambda I_1\left(\frac{1}{l}\right)I_0\left(\frac{\lambda}{l}\right)}{I_1^2\left(\frac{1}{l}\right)-I_0\left(\frac{1}{l}\right)I_2\left(\frac{1}{l}\right)}\cos\left(\frac{Z}{l}\right)\cos nT \quad \text{.. (4.4)}$$

and

$$P(\lambda, Z, T) = C - KZ\cos nT - \epsilon K\frac{I_1\left(\frac{1}{l}\right)I_0\left(\frac{\lambda}{l}\right)}{I_1^2\left(\frac{1}{l}\right)-I_0\left(\frac{1}{l}\right)I_2\left(\frac{1}{l}\right)}\cos\left(\frac{Z}{l}\right).\cos nT. \quad (4.5)$$

For $nT = 0$ the above results are in agreement with Khamrui (1963).

5. Numerical Discussion

The axial and radial velocity profiles for particular values of $\epsilon = 0{\cdot}1$ and $l = 1$ have been shown in Figs. 1 to 4 at different cross-sections of the

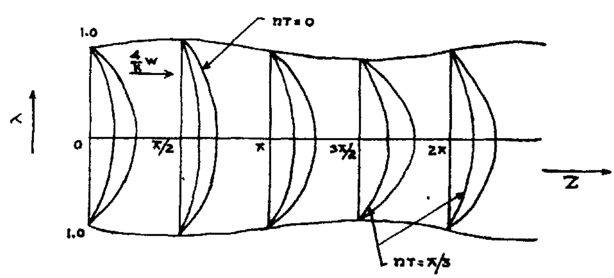

FIG. 1. The axial velocity profiles at different sections of a roughness wave for $\epsilon = 0{\cdot}1$ and $nT = 0$ and $\pi/3$.

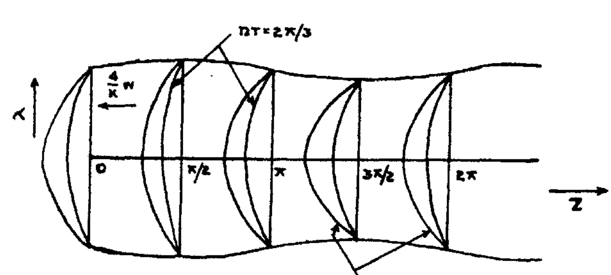

FIG. 2. The axial velocity profiles at different sections of a roughness wave for $\epsilon = 0{\cdot}1$ and $nT = 2\pi/3$ and π.

tube for various values of nT. As the width of the tube increases the magnitude of the axial velocity decreases and vice versa. It is interesting to note that the radial velocity profiles have a point of inflexion on the mid-plane.

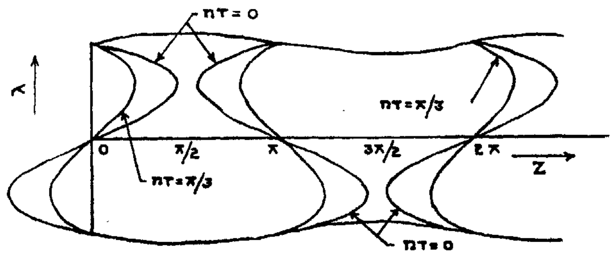

FIG. 3. The radial velocity profiles at different sections of a roughness wave for $\epsilon = 0 \cdot 1$ and $nT = 0$ and $\pi/3$.

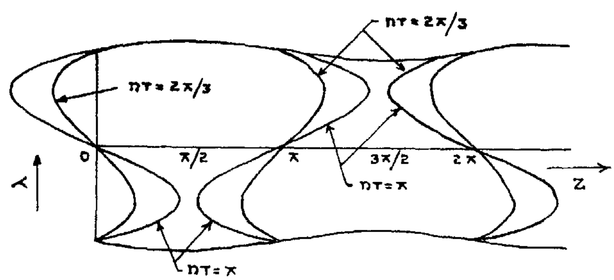

FIG. 4. The radial velocity profiles at different sections of a roughness wave for $\epsilon = 0 \cdot 1$ and $nT = 2\pi/3$ and π.

Thus the resulting effect of the axial and radial velocity profiles is that the direction of the flow is towards the walls if the width of the tube increases and away from the walls if the width of the tube decreases. The radial velocity which is developed due to the roughness of the walls is reduced to zero on the axis of the tube.

ACKNOWLEDGEMENT

The authors are extremely grateful to Prof. P. L. Bhatnagar for his encouragement throughout the preparation of this paper.

REFERENCES

Bhatnagar, P. L., and Mohan Rao, D. K. (1965). Flow of Reiner-Rivlin fluid between two concentric, rough circular cylinders rotating about their common axis. *Proc. Indian Acad. Sci.*, **62**, 347.

Bhatnagar, R. K., and Mathur, M. N. (1967). Heat transfer in a steady flow of a non-Newtonian fluid between two wavy walls situated symmetrically about a mid-plane. *Indian J. Pure appl. Phys.*, **5**, 37.

Citron, J. S. (1962). Slow viscous flow between rotating concentric cylinders with axial roughness. *J. appl. Mech.*, **29**, 188.

Khamrui, S. R. (1963). Slow steady flow of a viscous liquid through a circular tube with axial roughness. *Indian J. Mech. Math.*, **1**, 18.

Mathur, M. N. (1967). Slow steady motion of an elastico-viscous fluid with heat transfer in a wavy channel. *J. Indian Inst. Sci.*, **49**, 71.

Sexl, Th. (1930). Uber den von E. G. Richardson entdeckten Annulareffekt. *Z. Phys.*, **61**, 349.

Sneddon, I. N. (1951). Fourier Transforms. McGraw-Hill Book Co., Inc., New York.

Uchida, S. (1956). The pulsating viscous flow superposed on the steady laminar motion of incompressible fluid in a circular pipe. *Z. angew Math. Phys.*, **7**, 403.

Verma, P. D. (1960). The pulsating viscous flow superposed on the steady laminar motion of incompressible fluid between two coaxial cylinders. *Proc. natn. Inst. Sci. India*, **26**, 447.