INTEGRAL EQUATION METHOD OF SOLVING TWO-DIMENSIONAL LAPLACE EQUATION—I

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This paper provides two methods of solving a two-dimensional Laplace equation numerically based upon Green's formula. The analytical result states that a harmonic function may be considered as potentials of surface distributions with piecewise continuous density or potentials of double distributions with piecewise continuous moments. The integral representation of these potentials are available but involve some difficulties from a computational point of view. These can, however, be managed. Two problems, analytic solutions of which are known, are solved by these techniques using different quadrature formulae. The results are compared with analytic solutions.

Introduction

Laplace equation is an important partial differential equation governing a large number of physical problems arising in many branches of physics, engineering and applied mathematics. For example, this equation is satisfied by the following functions: (i) The gravitational potential in regions not occupied by attracting matter; (ii) the electrostatic potential in a uniform dielectric, in the theory of electrostatics; (iii) the magnetic potential in free space, in the theory of magneto-statics; (iv) the electric potential, in the theory of thermal equilibrium of solids; (v) the velocity potential at points of a homogeneous liquid, moving irrotationally, in hydrodynamical problems; and (vi) principal stress and torsion stress functions in a uniform isotropic bar, etc.

A function u is said to be harmonic in a region if the function and its derivatives up to the second order are continuous and satisfy Laplace equation in this region. Three well-known boundary value problems associated with Laplace equation are as follows:

Given a bounded domain G and a continuous function f defined on the boundary of G, one has to find a function u harmonic in G and continuous in $\bar{G}$ where $\bar{G}$ is the closure of G such that, for all points Q of the boundary, $u(Q) = f(Q)$. This is known as Dirichlet problem. The condition that G be bounded can be dropped and the condition that f be continuous can be relaxed somewhat. The problem of determining a function u harmonic in G, continuous in $\bar{G}$ and satisfying $\partial u/\partial n = g(s)$ on C, boundary of G, where g is a continuous function on C, is known as Neumann problem. In this case an additional condition is to be satisfied, namely $\int_C g(s)\,ds = 0$. Finally, the problem of determining a function u harmonic in G, continuous in $\bar{G}$ and satisfying $\partial u/\partial n = g(s)$ on one part of the boundary and $u(Q) = f(Q)$ on the remaining part is called Cauchy problem.

In this paper a numerical solution only to the first problem, namely the Dirichlet problem for a finite region bounded by a simply connected closed curve, has been suggested. It may be remarked that analytical solutions can be obtained for relatively very few problems and recourse has to be taken only to numerical techniques. Numerical methods based upon finite differences were developed by Southwell (1946), Allen (1959) and others and are known as relaxation methods. This method is eminently suited when the boundaries are straight. For curved boundaries the method becomes difficult to apply. Another method was proposed by Bhargava (1959) and was published (Bhargava and Radhakrishna 1963) with some more suitable modifications. The method involved a potential of single layer and potential of double layer. Method based upon single layer was exemplified by Soral (1966) for a circular region. As earlier remarked one might use single layer distribution or double layer distribution alone. Also for numerical computation one can use a variety of quadrature formulae. A study of both these methods involving a single layer and a double layer is made in this paper and have been tested in connection with some specific problems, for which analytical solutions are known. It may also be seen that the methods involve Fredholm's equation of the first and second kind respectively.

We have divided the contents of this paper into four parts. In Part I the methods leading to Fredholm's equations are discussed and include some techniques for facilitating computational work. In Parts II and III the solution of two problems is tried by each of the two different methods indicated above using simple sets of boundary conditions, one involving an odd function and another an even function. In the last part the results obtained in the preceding sections are analysed.

Part I

As is well known, the fundamental solution of Laplace equation $\nabla^2\Phi = 0$ in two dimensions is

$$\Phi = \log\frac{1}{\rho} = \log\frac{1}{\sqrt{x^2+y^2}}$$

where ρ is the length of the radius vector from origin to the point under consideration, while in three and higher dimensions it is $\Phi = 1/\rho^{n-2}$, n indicating the number of dimensions. This is the potential generated by a unit mass situated at the origin. If we have a set of masses at various points, we merely add the corresponding expressions. If we have a continuous distribution of mass $\sigma(x, y, z)$ over a region D and r is the distance of the point (x, y, z) from (λ, η, μ), the corresponding potential at (λ, η, μ) is

$$\Phi(\lambda, \eta, \mu) = \iiint_D \frac{\sigma(x, y, z)}{r} \, dx\, dy\, dz$$

$$= \iiint_D \frac{\sigma(x, y, z), \, dx\, dy\, dz}{\surd\{(\lambda-x)^2+(\eta-y)^2+(\mu-z)^2\}}.$$

We may likewise consider potentials associated with mass distributions on surfaces, along lines, etc., in either two or three dimensions. For the purpose of the present paper, we shall confine our attention to a two-dimensional case only. The potential for distribution σ on a closed curve C in a plane is given by

$$\Phi(P) = \oint_C \sigma(q) \log 1/r \, dq$$

where q is any point on the boundary and P is any point within C, r is the distance of q from P, and dq is the length of the arc at q. Thus a harmonic function $\Phi(P)$ at any point P in a simply connected domain D bounded by a contour C may be represented by

$$\Phi(P) = -\oint_C \sigma(q) \log |q-P| \, dq. \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

We distinguish a point on the boundary by the small letter p and indicate the value attained by $\Phi(P)$ at p by $\Phi(p)$, thus

$$\Phi(p) = -\oint_C \sigma(q) \log |q-p| \, dq. \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

It appears at first sight that this is not the Fredholm equation in the usual sense because the kernel $\log |q-p|$ involves a discontinuity at $q = p$. However, since the double integral $\int_C \int_C \log^2 |q-p| \, dq\, dp$ is finite, the eqn. (2) is still a Fredholm equation (Miklin 1957).

Hence to solve Dirichlet problem one must find the value of $\sigma(q)$ from (2), which is Fredholm equation of first kind, and substitute in (1) to get finally the value of $\Phi(P)$. To obtain $\sigma(q)$, we proceed as follows. Let L be the length of the curve C. Taking any point as the zero point on the curve we may write (2) as

$$\Phi(p) = -\int_0^L f(q) \, dq, \quad \text{where } f(q) = \sigma(q) \log |q-p|. \qquad \ldots \quad \ldots \quad (3)$$

By a suitable substitution, we change the limits of this integral from 0 to L to -1 to $+1$, then it may be evaluated by a suitable quadrature formula, e.g. Gauss-Legendre, Lobatto or Trapezoidal rule, etc. Of course, for the last one the limits need not be from -1 to $+1$. Thus

$$\Phi(p) = -\sum_{j=1}^{N} w_j f(q_j)+E$$

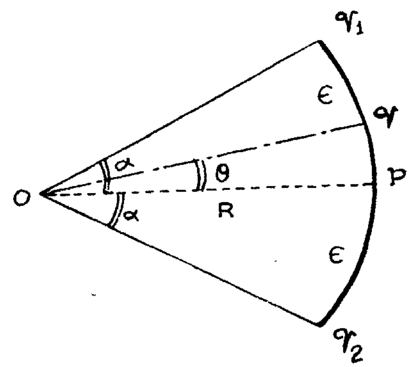
where q_j, w_j and E are called abscissae, weights and error respectively of the formula used. Tables (Stroud and Secrest 1966) are available for values of q_j and w_j for different values of N. Since p is an arbitrary fixed point on C, we allow it to take N values p_i, $i = 1, 2, \ldots, N$ and thus get a set of N-linear simultaneous equations in N unknowns, $\sigma(q_1)$, $\sigma(q_2)$, $\ldots$, $\sigma(q_N)$. As is the usual practice, we take E to be zero. So

$$\Phi(p_i) = -\sum_{j=1}^{N} w_j \sigma(q_j) \log |q_j-p_i|. \qquad (4)$$

The values of $\log |q-p|$ can be obtained from any mathematical table for all values of $|q-p|$ sufficiently greater than zero. But when q equals p, $\log |q-p|$ is undefined. However, in this case average value of $\log |q-p|$ may be computed as follows. If the arc length adjoining p is approximated to a straight line of length ϵ, then average value of $\log |q-p|$ is $\frac{1}{\epsilon}\int_0^\epsilon \log s\, ds = \log \epsilon - 1$.

To get a better approximation, the arc adjoining p may be approximated to the arc length ϵ of a circle of radius R, subtending an angle 2α at the centre. In this case the average value of $\log |q-p|$ is I_ϵ/ϵ where

$$I_\epsilon = \int_p^{q_1} \log |q-p|\, dq$$



$$= R\int_0^\alpha \log (2R \sin \theta/2)\, d\theta$$

$$= \epsilon(\log \epsilon-1)-(\epsilon^3/72R^2+\epsilon^5/14400R^4+\epsilon^7/1270080R^6) \text{ app.}$$

It may be seen here that if the curvature at the point p is very large, then the terms in the second bracket shall make substantial contribution. It can be proved that the coefficient matrix of $\sigma(q_i)$ in the R.H.S. of (4) is non-singular. Therefore, the system of linear simultaneous equations for $\sigma(q_i)$ in (4) may be solved by any of the well-known methods. Crouts method (Hildebrand

1956) has been used in the present problems. Having obtained $\sigma(q_i)$, the value of the potential at any point P may be found out from the formula

$$\Phi(P) = -\sum_{j=1}^{N} w_j \sigma(q_j) \log |q_j - P| \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

which is subject to some error, because the integral has been approximated by a sum of finite number of terms.

Two problems exemplifying the above techniques are given in Part II of this paper. The above method shall be termed as the 'first method' in Parts III and IV of this paper. We now give below the 'second method', which is based upon the double layer potential due to the distribution of moments.

Dirichlet problem in this case is solved by introducing a double layer potential

$$W(P) = \frac{1}{2\pi} \int_C \mu(q) \log' |q-P| \, dq \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

which yields the boundary equation

$$g(p) = \frac{1}{2\pi} \int_C \mu(q) \log' |q-p| \, dq + \tfrac{1}{2}\mu(p) \qquad \ldots \quad \ldots \quad (7)$$

where $W(p) \equiv g(p)$ on the boundary C and $\log' |q-p|$ is the outward normal derivative of $\log |q-p|$ at q. The term $\frac{1}{2}\mu(p)$ appears on the right-hand side because of the jump in the normal derivative as it crosses the boundary. Consequently, if the value of $g(p)$ is given on the boundary, the value of $\mu(q)$ can be found from (7), a Fredholm equation of second kind, in the same manner as in the first method. Having obtained $\mu(q)$, we shall substitute the values in (6) to obtain the harmonic function $W(P)$, which has the value $g(p)$ on the boundary.

The application of this formulation of integral equations involves the determination of the kernel $\log' |q-p|$, which can be evaluated as

$$\begin{aligned} \log' |q-p| &= \frac{\partial}{\partial \nu_q} \log |q-p| \\ &= \frac{\partial}{\partial x} \log |q-p| \cdot \frac{\partial x}{\partial \nu_q} + \frac{\partial}{\partial y} \log |q-p| \cdot \frac{\partial y}{\partial \nu_q} \\ &= \frac{(x-x_1)\cos\beta + (y-y_1)\sin\beta}{(x-x_1)^2+(y-y_1)^2} \qquad \ldots \quad \ldots \quad \ldots \quad (8) \end{aligned}$$

where (x, y) and (x_1, y_1) are the coordinates of q and p respectively. The angle of inclination of the outward normal at q to x-axis is denoted by β.

There is yet another form in which this formula can be used. By Cauchy-Riemann equations, it is well known that $\frac{\partial}{\partial \nu_q} \log |q-p| = \frac{\partial \theta}{\partial s}$, where θ is the angle which the radius vector $|q-p|$ makes with any line fixed in the plane, whence $\frac{\partial}{\partial \nu_q} \log |q-p|$ may be computed as the average change in θ as the point q moves a distance δs on the curve C, the point p remains fixed. It is proved later that the coefficient matrix in this case also will be non-singular.

Now, we introduce the three different quadrature formulae which have been used for solving integral equations in both the methods. The same two examples are solved by this method also in Part III to test the effectiveness of the method in contrast to the first one.

Gauss-Legendre Quadrature Formula

Let $f(x) \in C^{2n}[a, b]$ and $w(x) \geqslant 0$ be a weight function defined on $[a, b]$ with corresponding orthonormal polynomials $p_n(x)$. Let the zeros of $p_n(x)$ be $a < x_1 < x_2 < \ldots < x_n < b$, then we can find positive constants $w_1, w_2, \ldots, w_n$ such that

$$\int_a^b w(x) f(x)\, dx = \sum_{k=1}^{n} w_k f(x_k) + \frac{f^{(2n)}(\lambda)}{(2n)!\, k_n^2}, \quad a < \lambda < b.$$

It can be easily proved that

$$\sum_{k=1}^{n} w_k = a-b. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

In case $w(x) = 1$, $a = -1$, $b = +1$, the formula obtained is known as Gauss-Legendre quadrature formula. It is an open type quadrature formula and has degree of precision $(2r-1)$. It may be mentioned that if a quadrature formula yields exact results when $f(x)$ is an arbitrary polynomial of degree r or less, but fails to give exact results for at least one polynomial of degree $(r+1)$, it is said to possess a degree of precision equal to r (Hildebrand 1956).

Lobatto Quadrature Formula

Let

$$f(x) \in C^{2n-2}[-1, 1].$$

Then

$$\int_{-1}^{1} f(x)\, dx = \frac{2}{n(n-1)} [f(1)+f(-1)] + \sum_{j=2}^{n-1} w_j f(x_j) + E$$

where x_j is the $(j-1)$th zero of $P'_{n-1}(x)$, $P(x)$ is Legendre-polynomial and

$$E = E(f) = \frac{-n(n-1)^3\, 2^{2n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^3} f^{(2n-2)}(\lambda), \quad -1 < \lambda < 1.$$

Therefore, the degree of precision is $(2r-3)$. This formula is of a closed type and is very helpful specially when $f(x)$ displays peculiar behaviour at $x = \pm 1$, such as an apparent singularity, etc., or when $f(\pm 1) = 0$. In the latter case, the degree of precision is $(2r+1)$, whereas in Gauss formula, the use of r ordinates will still lead to a degree of precision $(2r-1)$.

Trapezoidal Rule

Under certain conditions, the trapezoidal rule gives surprisingly good results when it is applied to periodic functions—much better in fact than what might have been predicted from the error estimate. Denoting by Tn the n-point trapezoidal rule, the rule states:

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(a)+2f(a+h)+\ldots+2f(a+(n-1)h)+f(b)]+E_{Tn}$$

where $h = b-a/n$ and E_{Tn} is the error in the n-point trapezoidal rule (Tn). In case $f(x) \in C^{2k+1}[a, b]$ and $f(x)$ is a periodic function, then

$$|E_{Tn}| = \left| \int_a^b f(x)\,dx - Tn(f) \right| \leqslant \frac{C'}{n^{2k+1}}$$

where C' is a constant, independent of n (David and Robinowitz 1967). Under these conditions noting that for periodicity $f(a) = f(b)$ the trapezoidal rule takes the form

$$Tn = h \sum_{k=1}^{n} f(a+(k-1)h).$$

Consequently, if $p = b-a$, then

$$\int_a^b f(x)\,dx = \int_a^{a+p} f(x)\,dx = \int_0^p f(x)\,dx = \frac{p}{n}\sum_{k=1}^{n} f\left(\frac{(k-1)p}{n}\right). \quad \text{.. (10)}$$

Lastly, we prove an important result concerning matrices, which is required in the following parts of this paper. Let $A = [a_{ij}]$ be an arbitrary square matrix of order n. If

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad (i = 1, 2, \ldots, n) \quad \text{.. .. .. (11)}$$

then A is non-singular.

Suppose that A is singular and let $\vec{y} = (y_1, y_2, \ldots, y_n)$ be a non-zero vector satisfying the equation $A\vec{y} = 0$. Note that such a vector would exist for a singular matrix A. Let $|y_k| = \max_i |y_i|$, then taking the kth row

$$|a_{kk}||y_k| = |a_{kk}\,y_k| = \left|\sum_{j \neq k} a_{kj}\,y_j\right|$$

$$\leqslant \sum_{j \neq k} |a_{kj}||y_j| \leqslant |y_k| \sum_{j \neq k} |a_{kj}|$$

$$< |a_{kk}||y_k| \quad \text{from (11)}$$

which is impossible and, therefore, A is non-singular.

PART II

In this part we briefly mention the calculations done on Computer (IBM/7044) for two cases by first method for which the analytic solutions are known to check the usefulness and accuracy of the technique. On the boundary of a circle of radius $1/\pi$ (the factor $1/\pi$ is taken to avoid the constant repetition of π) the value of a function is x, i.e. $\Phi(p) = x$. For using Gauss-Legendre quadrature formula we put

$$q = s+1, \quad dq = ds \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

in (3) and get

$$\Phi(p) = -\int_{-1}^{1} \sigma(s+1) \log |s+1-p| \, ds.$$

Replacing p by p_i and substituting for $\Phi(p)$ in the above equation, we get

$$x_i = -\int_{-1}^{1} \sigma(s+1) \log |s+1-p_i| \, ds \qquad \ldots \quad \ldots \quad (13)$$

where p_i is the point (x_i, y_i)

$$= -\sum_{j=1}^{N} w_j \, \sigma(s_j+1) \log |s_j+1-p_i| \qquad \ldots \quad \ldots \quad (14)$$

where w_j and s_j are the weights and abscissae respectively given by Stroud and Secrest (1966).

Taking $p_i = (s_j+1)\delta_{ij}$; $i, j = 1, 2, \ldots, N$, in (14) we obtain a set of N-linear simultaneous equations and solve it for σ's.

Substituting the values from (12) in (1), we get

$$\Phi(P) = -\int_{-1}^{1} \sigma(s+1) \log |s+1-P| \, ds$$

$$= -\sum_{j=1}^{N} \sigma(s_j+1) \log |s_j+1-P| \, w_j, \quad \text{by Gauss-Legendre quadrature formula}$$

$$= -\tfrac{1}{2} \sum_{j=1}^{N} w_j \, \sigma(s_j+1) \log \{(x_j-\xi)^2+(y_j-\eta)^2\} \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

where (x_j, y_j) are the coordinates of the point on the curve corresponding to the arc-length (s_j+1) and (ξ, η) is the point P. Substituting in (15) the values of w_j and σ's obtained from (14), we finally get $\Phi(P)$. To test, $\Phi(P)$ was evaluated at eight points, one in each quadrant and two points on each axis, one on the positive side and the other on the negative side, inside the circle.

For Lobatto quadrature formula, only the weights and abscissae given by Stroud and Secrest (1966) are to be changed in eqns. (14) and (15).

Finally the trapezoidal rule can be applied more easily as follows:

Applying (10) in (3) and replacing p by p_i, the following equation is obtained:

$$\Phi(p_i) = -\frac{L}{N}\sum_{k=1}^{N} \sigma\left(\frac{L(k-1)}{N}\right) \log\left|\frac{L(k-1)}{N} - p_i\right|. \quad \ldots \quad \ldots \quad (16)$$

Putting $p_i = \frac{L(k-1)}{N}\,\delta_{ik}$, $i, k = 1, 2, \ldots, N$ we solve the above set of N-linear simultaneous equations to obtain σ's.

Further (1) can be written as

$$\Phi(P) = -\int_0^L \sigma(q) \log|q-P|\,dq$$

$$= -\frac{L}{N}\sum_{k=1}^{N} \sigma\left(\frac{L(k-1)}{N}\right) \log\left|\frac{L(k-1)}{N} - P\right|, \quad \text{by trapezoidal rule}$$

$$= -\frac{L}{2N}\sum_{k=1}^{N} \sigma\left(\frac{L(k-1)}{N}\right) \log\{(x_k-\xi)^2+(y_k-\eta)\}^2 \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

where (x_k, y_k) are the coordinates of the point of the curve corresponding to the length $L(k-1)/N$ and (ξ, η) are the coordinates of the point P, inside C. Substituting σ's obtained from (16) in (17), we get $\Phi(P)$ at the same eight points.

Values of σ's were obtained from (14) where w_j and s_j refer to the weights and abscissae in Gauss' quadrature formula, and also when w_j and s_j refer to those in Lobatto formula, for $N = 8$, 16, 24 and 32. Values of σ's were also obtained from (16), where trapezoidal rule is used, for the values of N. These were compared with their corresponding analytic values. In the first two cases, the results were found to be better than the last one, which happens ultimately with the values of $\Phi(P)$ also. In Table I computed values of $\Phi(P)$ by all the three quadrature formulae as well as their analytic values are shown.

Similar calculations were done for another case where $\Phi(p)$ was taken to be equal to x^2-y^2 and the results are given in Table II. It turns out that in these cases where one function ($\Phi = x$) was taken to be odd and another function ($\Phi = x^2-y^2$) even, the Gauss-Legendre quadrature formula gives better results.

Part III

In this part we mention the computations done by the second method on the computer indicated in Part I of this paper. The same problems were done. We shall briefly describe the method that is followed and results obtained by each of the quadrature formula.

First of all, it can be easily proved that $\frac{\partial}{\partial n_q} \log|q-p| = \frac{1}{2r}$ for a circle of radius r by taking the points p and q as (r, ψ) and (r, θ) respectively and also

Table I

Values of $\Phi(P)$

Coordinates of the point P		Analytic value	Computed values					
X	Y		Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
					$\Phi(p) = x,\ N = 8$			
0·2122	0·0000	0·2122	0·20471242	0·00748758	0·19676222	0·01543778	0·19928469	0·01291531
0·1061	0·1061	0·1061	0·11383563	0·00773563	0·11716227	0·01106227	0·10123878	0·00486122
0·0000	0·2122	0·0000	−0·00172520	0·00172520	−0·00502260	0·00502260	−0·00356693	0·00356693
−0·1061	0·1061	−0·1061	−0·10670039	0·00060039	−0·10370826	0·00239174	−0·11203414	0·00593414
−0·2122	0·0000	−0·2122	−0·21519096	0·00299096	−0·20688226	0·00531774	−0·19928469	0·01291531
−0·1061	−0·1061	−0·1061	−0·10670039	0·00060039	−0·10370826	0·00239174	−0·10123878	0·00486122
0·0000	−0·2122	0·0000	−0·00172520	0·00172520	−0·00502261	0·00502261	0·00356692	0·00356692
0·1061	−0·1061	0·1061	0·11383562	0·00773562	0·11716227	0·01106227	0·11203414	0·00593414
					$\Phi(p) = x,\ N = 16$			
0·2122	0·0000	0·2122	0·21688898	0·00468898	0·21681315	0·00461315	0·20540890	0·00679110
0·1061	0·1061	0·1061	0·10938337	0·00328337	0·10957908	0·00347908	0·10223900	0·00386100
0·0000	0·2122	0·0000	0·00096249	0·00096249	0·00090609	0·00090609	−0·00254110	0·00254110
−0·1061	0·1061	−0·1061	−0·10693471	0·00083471	−0·10687829	0·00077829	−0·11004487	0·00394487
−0·2122	0·0000	−0·2122	−0·21412672	0·00192672	−0·21387942	0·00167942	−0·20540890	0·00679110
−0·1061	−0·1061	−0·1061	−0·10693471	0·00083471	−0·10687829	0·00077829	−0·10223901	0·00386099
0·0000	−0·2122	0·0000	0·00096249	0·00096249	0·00090609	0·00090609	0·00254109	0·00254109
0·1061	−0·1061	0·1061	0·10938337	0·00328337	0·10957908	0·00347908	0·11004487	0·00394487
					$\Phi(p) = x,\ N = 24$			
0·2122	0·0000	0·2122	0·21611556	0·00391556	0·21624946	0·00404946	0·20783088	0·00436912
0·1061	0·1061	0·1061	0·10837860	0·00227860	0·10847791	0·00237791	0·10302896	0·00307104
0·0000	0·2122	0·0000	0·00046459	0·00046459	0·00050778	0·00050778	−0·00189126	0·00189126
−0·1061	0·1061	−0·1061	−0·10671389	0·00061389	−0·10672240	0·00062240	−0·10875237	0·00265237
−0·2122	0·0000	−0·2122	−0·21353863	0·00133863	−0·21353494	0·00133494	−0·20782089	0·00436911
−0·1061	−0·1061	−0·1061	−0·10671389	0·00061389	−0·10672240	0·00062240	−0·10302896	0·00307104
0·0000	−0·2122	0·0000	−0·00046459	0·00046459	0·00050777	0·00050777	−0·00189125	0·00189125
0·1061	−0·1061	0·1061	0·10837861	0·00227861	0·10848891	0·00237791	0·10875238	0·00265238
					$\Phi(p) = x,\ N = 32$			
0·2122	0·0000	0·2122	0·21524597	0·00304597	0·21533914	0·00313914	0·20886510	0·00333490
0·1061	0·1061	0·1061	0·10783164	0·00173164	0·10788752	0·00178752	0·10354489	0·00255511
0·0000	0·2122	0·0000	0·00033870	0·00033870	0·00035132	0·00035132	−0·00150127	0·00150127
−0·1061	0·1061	−0·1061	−0·10657575	0·00047575	−0·10658622	0·00048622	−0·10806035	0·00196035
−0·2122	0·0000	−0·2122	−0·21321992	0·00101992	−0·21323570	0·00103570	−0·10886510	0·00333490
−0·1061	−0·1061	−0·1061	−0·10657576	0·00047576	−0·10658622	0·00048622	−0·10354490	0·00255510
0·0000	−0·2122	0·0000	0·00033870	0·00033870	0·00035132	0·00035132	0·00150127	0·00150127
0·1061	−0·1061	0·1061	0·10783164	0·00173164	0·10788753	0·00178753	0·10806036	0·00196036

TABLE II

Values of $\Phi(P)$

Coordinates of the point P		Analytic value	Computed values					
X	Y		Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
			$\Phi(p) = x^2-y^2,\ N = 8$					
0·2122	0·0000	0·04502883	0·02861544	0·01641339	0·01844908	0·02657975	0·03023010	0·01479873
0·1061	0·1061	0·00000000	0·00297520	0·00297520	0·00255872	0·00255872	−0·01051674	0·01051674
0·0000	0·2122	−0·04502883	−0·05282685	0·00779802	−0·03993867	0·00509016	−0·06006643	0·01503760
−0·1061	0·1061	0·00000000	−0·00059084	0·00059084	−0·00407635	0·00407635	−0·00947558	0·00947558
−0·2122	0·0000	0·04502883	0·04513605	0·00010722	0·03799085	0·00703798	0·03023010	0·01479873
−0·1061	−0·1061	0·00000000	−0·00059084	0·00059084	−0·00407635	0·00407635	−0·01051674	0·01051674
0·0000	−0·2122	−0·04502883	−0·05282685	0·00779802	0·00255872	0·04247011	−0·06006643	0·01503760
0·1061	−0·1061	0·00000000	0·00297520	0·00297520	−0·03993867	0·03993867	−0·00947558	0·00947558
			$\Phi(p) = x^2-y^2,\ N = 16$					
0·2122	0·0000	0·04502883	0·04585153	0·00082270	0·04546563	0·00043680	0·03848173	0·00654710
0·1061	0·1061	0·00000000	−0·00008027	0·00008027	−0·00012891	0·00012891	−0·00473142	0·00473142
0·0000	0·2122	−0·04502883	−0·04696237	0·00193354	−0·04749385	0·00246502	−0·05038008	0·00535125
−0·1061	0·1061	0·00000000	−0·00052419	0·00052419	−0·00062139	0·00062139	−0·00332281	0·00332281
−0·2122	0·0000	0·04502883	0·04550470	0·00047587	0·04528405	0·00025522	0·03848173	0·00654710
−0·1061	−0·1061	0·00000000	−0·00052419	0·00052419	−0·00062139	0·00062139	−0·00473142	0·00473142
0·0000	−0·2122	−0·04502883	−0·04696237	0·00193354	−0·04749386	0·00246503	−0·05038008	0·00535125
0·1061	−0·1061	0·00000000	−0·00008027	0·00008027	−0·00012892	0·00012892	−0·00332281	0·00332281
			$\Phi(p) = x^2-y^2,\ N = 24$					
0·2122	0·0000	0·04502883	0·04641649	0·00138766	0·04643008	0·00140125	0·04106105	0·00396778
0·1061	0·1061	0·00000000	−0·00002336	0·00002336	−0·00002772	0·00002772	−0·00310132	0·00310132
0·0000	0·2122	−0·04502883	−0·04640837	0·00137954	−0·04646284	0·00143401	−0·04836747	0·00333864
−0·1061	0·1061	0·00000000	−0·00035832	0·00035832	−0·00038384	0·00038384	−0·00204826	0·00204826
−0·2122	0·0000	0·04502883	0·04539116	0·00036233	0·04536546	0·00033663	0·04106105	0·00396778
−0·1061	−0·1061	0·00000000	−0·00035832	0·00035832	−0·00038385	0·00038385	−0·00310133	0·00310133
0·0000	−0·2122	−0·04502883	−0·04640837	0·00137954	−0·04646284	0·00143401	−0·04836748	0·00333865
0·1061	−0·1061	0·00000000	−0·00002336	0·00002336	−0·00002772	0·00002772	−0·00204826	0·00204826
			$\Phi(p) = x^2-y^2,\ N = 32$					
0·2122	0·0000	0·04502883	0·04619621	0·00116738	0·04622509	0·00119626	0·04211479	0·00291404
0·1061	0·1061	0·00000000	−0·00001580	0·00001580	−0·00001739	0·00001739	−0·00235066	0·00235066
0·0000	0·2122	−0·04502883	−0·04613385	0·00110502	−0·04616451	0·00113568	−0·04746068	0·00243185
−0·1061	0·1061	0·00000000	−0·00027095	0·00027095	−0·00028247	0·00028247	−0·00151206	0·00151206
−0·2122	0·0000	0·04502883	0·04531281	0·00028398	0·04531043	0·00028160	0·04211479	0·00291404
−0·1061	−0·1061	0·00000000	−0·00027095	0·00027095	−0·00028248	0·00028248	−0·00235066	0·00235066
0·0000	−0·2122	−0·04502883	−0·04613386	0·00110503	−0·04616452	0·00113569	−0·04746068	0·00243185
0·1061	−0·1061	0·00000000	−0·00001580	0·00001580	−0·00001739	0·00001739	−0·00151207	0·00151207

$\dfrac{\partial}{\partial n_q} = \left(\dfrac{\partial}{\partial r}\right)_{(r,\,\theta)}$. By substituting the value of $\log' |q-p|$ and observing that $ds = r\,d\theta$ in (7), we obtain

$$g(p) = \frac{1}{4\pi}\int_0^{2\pi} \mu(r\cos\theta,\, r\sin\theta)\,d\theta + \tfrac{1}{2}\mu(p). \qquad \ldots \qquad \ldots \quad (18)$$

We change the limits of the integral in (18) from $(0, 2\pi)$ to $(-1, 1)$ by

$$\theta = \pi\Phi+\pi, \quad d\theta = \pi\,d\Phi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

obtaining

$$g(p) = \tfrac{1}{4}\int_{-1}^{1} \mu(-r\cos\pi\Phi,\, -r\sin\pi\Phi)\,d\Phi + \tfrac{1}{2}\mu(p). \quad \ldots \quad \ldots \quad (20)$$

Now by using Gauss-Legendre quadrature formula and replacing p by p_i

$$g(p_i) = \tfrac{1}{4}\sum_{k=1}^{N} w_k\,\mu(-r\cos\pi\Phi_k,\, -r\sin\pi\Phi_k) + \tfrac{1}{2}\mu(p_i).$$

Writing $\mu(-r\cos\pi\Phi_k, -r\sin\pi\Phi_k) = \mu_k$ and $p_i = (-r\cos\pi\Phi_i, -r\sin\pi\Phi_i)$, $i = 1, 2, \ldots, N$

$$g(p_i) = \tfrac{1}{4}\sum_{\substack{k=1\\k\neq i}}^{N} w_k\,\mu_k + (\tfrac{1}{4}w_i + \tfrac{1}{2})\mu_i$$

or

$$-4r\cos\pi\Phi_i = \sum_{\substack{k=1\\k\neq i}}^{N} w_k\,\mu_k + (w_i+2)\mu_i\,; \quad \text{since } g(p_i) = x_i\,. \quad \ldots \quad (21)$$

Thus (21) represents a set of N-linear simultaneous equations with coefficient matrix

$$M = \begin{bmatrix} (w_1+2) & w_2 & \ldots & w_N \\ w_1 & (w_2+2) & \ldots & w_N \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_1 & w_2 & \ldots & (w_N+2) \end{bmatrix}.$$

From (9)

$$\sum_{k=1}^{N} w_k = 2 \quad \text{or} \quad \sum_{\substack{k=1\\k\neq i}}^{N} w_k + w_i = 2.$$

Therefore

$$(2+w_i) > \sum_{\substack{k=1\\k\neq i}}^{N} w_k, \quad \text{since } w_i > 0, \quad i = 1, 2, \ldots, N.$$

Consequently, from the result proved in (11), M is non-singular.

So, we can solve (21) for μ_i, $i = 1, 2, \ldots, N$. Now to get $W(P)$, we write (6) as

$$W(P) = \frac{r}{2\pi}\int_0^{2\pi} \mu(r\cos\theta, r\sin\theta)\{r-R\cos(\theta-\psi)\}\{r^2+R^2-2rR\cos(\theta-\psi)\}^{-1}\,d\theta \quad \ldots (22)$$

where (r, θ) and (R, ψ), $R < 1/\pi$ are the coordinates of the points q and P respectively. After changing the limits of this integral by the substitution (19) to $(-1, 1)$, we apply Gauss-Legendre quadrature formula to obtain

$$W(P) = \frac{r}{2}\sum_{k=1}^{N} w_k\,\mu_k\{r+R\cos(\pi\Phi_k-\psi)\}\{r^2+R^2+2r\,R\cos(\pi\Phi_k-\psi)\}^{-1}. \quad (23)$$

Thus, substituting μ_k obtained from (21) and w_k, Φ_k from the tables given by Stroud and Secrest (1966), we get $W(P)$ at all those eight points mentioned in Part II.

For applying Lobatto quadrature formula, we have to change weights and abscissae, i.e. w_k and Φ_k only, in (21) and (23) according to the tables given by Stroud and Secrest (1966) and then we obtain μ_k from the former equation to substitute in the latter to get $W(P)$.

For the trapezoidal rule, we proceed as follows. Applying this rule in the form given in (10) into (18), we get

$$g(p) = \frac{1}{2N}\sum_{k=1}^{N}\mu_k+\tfrac{1}{2}\mu(p), \quad \text{where } \mu_k = \mu\left(r\cos\frac{2\pi(k-1)}{N}, r\sin\frac{2\pi(k-1)}{N}\right).$$

Replacing in this equation p by p_i, where

$$p_i = \left(r\cos\frac{2\pi(i-1)}{N}, r\sin\frac{2\pi(i-1)}{N}\right), \quad i = 1, 2, \ldots, N.$$

$$g(p_i) = \frac{1}{2N}\sum_{\substack{k=1\\k\neq i}}^{N}\mu_k+\frac{1}{2N}(N+1)\mu_i$$

or

$$2Nr\cos\left(\frac{2\pi(i-1)}{N}\right) = \sum_{\substack{k=1\\k\neq i}}^{N}\mu_k+(N+1)\mu_i, \quad \text{since } g(p_i) = x_i. \quad \ldots (24)$$

Thus, the set of N-linear simultaneous equations represented by (24) has the coefficient matrix

$$Z = \begin{bmatrix} (N+1) & 1 & \ldots & 1 \\ 1 & (N+1) & \ldots & 1 \\ . & . & & . \\ . & . & & . \\ 1 & 1 & \ldots & (N+1) \end{bmatrix}$$

Table III

Values of W(P)

Coordinates of the point P		Analytic value	Computed values					
X	Y		Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
			$g(p) = x, \ N = 8$					
0·2122	0·0000	0·2122	0·21296054	0·00076054	0·20713304	0·00506696	0·24019559	0·02799559
0·1061	0·1061	0·1061	0·10580904	0·00029096	0·10133982	0·00476018	0·10752626	0·00142626
0·0000	0·2122	0·0000	0·01403386	0·01403386	0·02780355	0·02780355	0·00000000	0·00000000
−0·1061	0·1061	−0·1061	−0·11574287	0·00964287	−0·12467177	0·01857177	−0·10752627	0·00142627
−0·2122	0·0000	−0·2122	−0·14290320	0·06929680	−0·12386503	0·08833497	−0·24019560	0·02799560
−0·1061	−0·1061	−0·1061	−0·11574286	0·00964286	−0·12467178	0·01857178	−0·10752627	0·00142627
0·0000	−0·2122	0·0000	0·01403384	0·01403384	0·02780353	0·02780353	−0·00000001	0·00000001
0·1061	−0·1061	0·1061	0·10580903	0·00029097	0·10133981	0·00476019	0·10752626	0·00142626
			$g(p) = x, \ N = 16$					
0·2122	0·0000	0·2122	0·21220640	0·00000640	0·21218080	0·00001920	0·21325105	0·00105105
0·1061	0·1061	0·1061	0·10609922	0·00000078	0·10607733	0·00002267	0·10610346	0·00000346
0·0000	0·2122	0·0000	−0·00197020	0·00197020	−0·00083002	0·00083002	0·00000000	0·00000000
−0·1061	0·1061	−0·1061	−0·10603029	0·00006971	−0·10583412	0·00026588	−0·10610346	0·00000346
−0·2122	0·0000	−0·2122	−0·20249697	0·00970303	−0·19959964	0·01260036	−0·21325106	0·00105106
−0·1061	−0·1061	−0·1061	−0·10603029	0·00006971	−0·10583412	0·00026588	−0·10610347	0·00000347
0·0000	−0·2122	0·0000	−0·00197022	0·00197022	−0·00083003	0·00083003	−0·00000001	0·00000001
0·1061	−0·1061	0·1061	0·10609922	0·00000078	0·10607732	0·00002268	0·10610346	0·00000346
			$g(p) = x, \ N = 24$					
0·2122	0·0000	0·2122	0·21220000	0·00000000	0·21219993	0·00000007	0·21224091	0·00004091
0·1061	0·1061	0·1061	0·10609999	0·00000001	0·10609987	0·00000013	0·10609999	0·00000001
0·0000	0·2122	0·0000	0·00004148	0·00004148	−0·00018860	0·00018860	−0·00000000	0·00000000
−0·1061	0·1061	−0·1061	−0·10609809	0·00000191	−0·10610034	0·00000034	−0·10610000	0·00000000
−0·2122	0·0000	−0·2122	−0·21094650	0·00125350	−0·21057059	0·00162941	−0·21224091	0·00004091
−0·1061	−0·1061	−0·1061	−0·10609809	0·00000191	−0·10610033	0·00000033	−0·10610001	0·00000001
0·0000	−0·2122	0·0000	0·00004146	0·00004146	−0·00018862	0·00018862	−0·00000001	0·00000001
0·1061	−0·1061	0·1061	0·10609997	0·00000003	0·10609986	0·00000014	0·10609999	0·00000001
			$g(p) = x, \ N = 32$					
0·2122	0·0000	0·2122	0·21219996	0·00000004	0·21219996	0·00000004	0·21220157	0·00000157
0·1061	0·1061	0·1061	0·10609998	0·00000002	0·10609998	0·00000002	0·10609998	0·00000002
0·0000	0·2122	0·0000	0·00001407	0·00001407	0·00002187	0·00002187	0·00000000	0·00000000
−0·1061	0·1061	−0·1061	−0·10610008	0·00000008	−0·10610010	0·00000010	−0·10609998	0·00000002
−0·2122	0·0000	−0·2122	−0·21203985	0·00016015	−0·21199200	0·00020800	−0·21220156	0·00000156
−0·1061	−0·1061	−0·1061	−0·10610009	0·00000009	−0·10610010	0·00000010	−0·10610000	0·00000000
0·0000	−0·2122	0·0000	0·00001406	0·00001406	0·00002186	0·00002186	0·00000000	0·00000000
0·1061	−0·1061	0·1061	0·10609998	0·00000002	0·10609997	0·00000003	0·10609998	0·00000002

TABLE IV

Values of $W(P)$

Coordinates of the point P		Analytic value	Computed values					
X	Y		Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
					$g(p) = x^2-y^2,\ N = 8$			
0·2122	0·0000	0·04502883	0·04532990	0·00030107	0·04302296	0·00200587	0·05611115	0·01108232
0·1061	0·1061	0·00000000	0·00112405	0·00112405	−0·00219352	0·00219352	0·00000000	0·00000000
0·0000	0·2122	−0·04502883	−0·06351335	0·01848452	−0·03753832	0·00749051	−0·05611114	0·01108231
−0·1061	0·1061	0·00000000	0·00589734	0·00589734	0·00584518	0·00584518	0·00000000	0·00000000
−0·2122	0·0000	0·04502883	0·01759706	0·02743177	0·01005942	0·03496941	0·05611114	0·01108231
−0·1061	−0·1061	0·00000000	0·00589735	0·00589735	0·00584518	0·00584518	0·00000000	0·00000000
0·0000	−0·2122	−0·04502883	−0·06351334	0·01848451	−0·03753831	0·00749052	−0·05611114	0·01108231
0·1061	−0·1061	0·00000000	0·00112405	0·00112405	0·00219352	0·00219352	0·00000000	0·00000000
					$g(p) = x^2-y^2,\ N = 16$			
0·2122	0·0000	0·04502883	0·04503137	0·00000254	0·04502124	0·00000759	0·04544491	0·00041608
0·1061	0·1061	0·00000000	0·00000570	0·00000570	−0·00000915	0·00000915	0·00000000	0·00000000
0·0000	0·2122	−0·04502883	−0·04541993	0·00039110	−0·04773108	0·00270225	−0·04544491	0·00041608
−0·1061	0·1061	0·00000000	−0·00013213	0·00013213	−0·00021738	0·00021738	0·00000000	0·00000000
−0·2122	0·0000	0·04502883	0·04118781	0·00384102	0·04004087	0·00498796	+0·04544491	0·00041608
−0·1061	−0·1061	0·00000000	−0·00013213	0·00013213	−0·00021738	0·00021738	0·00000000	0·00000000
0·0000	−0·2122	−0·04502883	−0·04541993	0·00039110	−0·04773108	0·00270225	−0·04544491	0·00041608
0·1061	−0·1061	0·00000000	0·00000570	0·00000570	−0·00000916	0·00000916	0·00000000	0·00000000
					$g(p) = x^2-y^2,\ N = 24$			
0·2122	0·0000	0·04502883	0·04502884	0·00000001	0·04502881	0·00000002	0·04504504	0·00001621
0·1061	0·1061	0·00000000	0·00000002	0·00000002	−0·00000007	0·00000007	0·00000000	0·00000000
0·0000	0·2122	−0·04502883	−0·04484032	0·00018851	−0·04485286	0·00017597	−0·04504504	0·00001621
−0·1061	0·1061	0·00000000	0·00000142	0·00000142	0·00000390	0·00000390	0·00000000	0·00000000
−0·2122	0·0000	0·04502883	0·04453262	0·00049621	0·04438382	0·00064501	0·04504503	0·00001620
−0·1061	−0·1061	0·00000000	0·00000142	0·00000142	0·00000390	0·00000390	0·00000000	0·00000000
0·0000	−0·2122	−0·04502883	−0·04484031	0·00018852	−0·04485286	0·00017597	−0·04504504	0·00001621
0·1061	−0·1061	0·00000000	0·00000002	0·00000002	−0·00000004	0·00000004	0·00000000	0·00000000
					$g(p) = x^2-y^2,\ N = 32$			
0·2122	0·0000	0·04502883	0·04502883	0·00000000	0·04502883	0·00000000	0·04502946	0·00000063
0·1061	0·1061	0·00000000	0·00000000	0·00000000	0·00000000	0·00000000	0·00000000	0·00000000
0·0000	0·2122	−0·04502883	−0·04503974	0·00001091	−0·04501912	0·00000971	−0·04502946	0·00000063
−0·1061	0·1061	0·00000000	0·00000001	0·00000001	−0·00000002	0·00000002	0·00000000	0·00000000
−0·2122	0·0000	0·04502883	0·04496544	0·00006339	0·04494650	0·00008233	0·04502946	0·00000063
−0·1061	−0·1061	0·00000000	0·00000001	0·00000001	−0·00000002	0·00000002	0·00000000	0·00000000
0·0000	−0·2122	−0·04502883	−0·04503973	0·00001090	−0·04501912	0·00000971	−0·04502946	0·00000063
0·1061	−0·1061	0·00000000	0·00000000	0·00000000	0·00000000	0·00000000	0·00000000	0·00000000

which is non-singular, for the same reasons given earlier and therefore we can solve (24) to get μ's. Using (10) into (22) we obtain

$$W(P) = \frac{r}{N} \sum_{k=1}^{N} \mu_k \left\{ r - R \cos\left(\frac{2\pi(k-1)}{N} - 1\right) \right\} \left\{ r^2 + R^2 - 2rR \cos\left(\frac{2\pi(k-1)}{N} - \psi\right) \right\}^{-1}. \quad \ldots (25)$$

Substituting μ_k, $k = 1, 2, \ldots, N$, obtained from (24), we finally get $W(P)$ at all those points.

Values of μ's obtained from (21) by Gauss-Legendre and Lobatto quadrature formulae and from (24) by trapezoidal rule were compared with their analytic values and were found identical up to six decimal figures. Analytic and computed values of $W(P)$ by all the three quadrature formulae for different values of N are given in Table III.

For the second problem, where $g(p) = x^2 - y^2$, the procedure is the same except for a few changes, e.g. in equation (21) we have to replace $g(p_i)$ by $2r^2 \cos 2\pi\Phi_i$ and in (22) by $r^2 \cos\left(\frac{4\pi(i-1)}{N}\right)$, etc. Values of $W(P)$ were computed by all the three quadrature formulae and are shown in Table IV. From the results, it seems that trapezoidal rule in both cases is most suitable.

Part IV

Looking into the tables, it may be seen that for the first method, Gauss-Legendre quadrature formula provides better results. While, with the second method, it is the trapezoidal rule which gives surprisingly accurate results. Of course, for a smaller value of N, say $N = 8$, it is not very impressive but immediately afterwards it converges and, for $N = 32$, the error is about one in a million. The reason for this is as follows:

The integral involved in the second method, to which we applied trapezoidal rule, is

$$\int_0^{2\pi} \mu(r \cos\theta, r \sin\theta)\, d\theta, \quad \text{where } r = 1/\pi.$$

The integrand is a periodic function of $\sin\theta$ and $\cos\theta$, of period 2π. Consequently, since $\mu(0) = \mu(2\pi)$, the formula (10) was applied. Now consider the error

$$E_{Tn}(f) = \frac{p}{n} \sum_{k=1}^{n} f\left(\frac{(k-1)p}{n}\right) - \int_0^p f(x)\, dx.$$

It is easily verified that

$$E_{Tn}\left(e^{\frac{i2\pi jx}{p}}\right) = \begin{cases} p, & j \neq 0,\ n/j \\ 0, & \text{otherwise} \end{cases}, \quad i = \sqrt{-1}.$$

This means that the trapezoidal rule (Tn) is exact for the $2n$ periodic functions 1, $\sin x$, $\cos x, \ldots, \sin(n-1)x$, $\cos(n-1)x$, $\sin nx$. And therefore in this case we find no error except for that one which is induced in solving the set of linear simultaneous equations. The reason why Gauss-Legendre and Lobatto quadrature formulae give error is that μ has derivatives of all orders.

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