ON THREE PART MIXED BOUNDARY VALUE PROBLEMS OF ELASTOSTATICS FOR AN ISOTROPIC CYLINDER CONTAINING TWO STRIP CRACKS OPENED BY INTERNAL PRESSURE

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We consider the problem of determining the distribution of stress in a long isotropic elastic cylinder containing two strip cracks situated symmetrically on a diametral plane. Two types of problems are considered. Firstly, we assume that the lateral surface of the cylinder is stress-free while secondly, the curved surface of the cylinder is supposed to be rigidly clamped. In both the problems the cracks are opened by a constant internal pressure p_0. We first reduce the problems to that of solving a set of triple integral equations with cosine kernel. These equations are solved by using the finite Hilbert transform technique. Analytical expressions for stress intensity factors, shape of the deformed cracks and the energy required to open the cracks are derived.

1. INTRODUCTION

Recently Srivastava and Lowengrub (1970) have developed a finite Hilbert transform technique for solving triple integral equations involving trigonometric kernels. This technique has proved very effective in the study of two-dimensional three-part mixed boundary value problems (Lowengrub and Srivastava 1968*a*, *b*; Srivastava 1970). Here, we shall apply it to the problem of determining the distribution of stress in a long isotropic elastic cylinder containing two coplanar strip cracks situated in a diametral plane parallel to the generators of the cylinder. The material of the cylinder is supposed to be homogeneous and isotropic with modulus of rigidity $\mu = Y/2(1+\eta)$ where Y is the Young's modulus and η Poisson's ratio. Recently Srivastav and Narain (1968) have solved the problem of finding distribution of stress in a right circular cylinder with a strip crack lying in the diametral plane. They have derived expressions for the components of displacement vector and stress tensor. We shall use these expressions and show that in the case of two coplanar strip cracks these expressions lead to a set of triple integral equations. By using the finite Hilbert transform technique developed by Srivastava and Lowengrub (1970), we are able to reduce these triple integral equations to a single Fredholm integral equation which can be solved by the well-known iterative procedure.

We consider the case in which the radius of cylinder is ρ and the distances of the two edges of the cracks from the centre of a diametral plane are a and b respectively. In section 2 boundary conditions appropriate to two types of problems are formulated. In section 3 we quote the expressions for the components of displacement vector and stress tensor from the paper of Srivastav and Narain (1968). In section 4 we list some results which we shall use in future sections. In sections 5 and 6 the problems have been reduced to that of solving a Fredholm integral equation of second kind. A solution of this basic Fredholm integral equation for $\rho \gg b$, up to the order (ρ^{-8}), is derived. The remaining sections deal with the calculation of stress intensity factors at the two ends of the crack and the energy required to open the cracks.

2. Formulation of Problems

We shall consider displacement field in a perfectly elastic cylinder containing two strip cracks lying symmetrically in a diametral plane parallel to the generators of the cylinder. The material of cylinder is supposed to be homogeneous and isotropic with modulus of rigidity μ and Poisson's ratio η. The two-dimensional problem is discussed under the assumptions of plane strain within the classical theory of infinitesimal elasticity. The cylinder is supposed to occupy the region $0 \leqslant r \leqslant \rho$, $0 \leqslant \theta \leqslant 2\pi$, $-\infty < z < \infty$ (using cylindrical coordinates), with the axis of the cylinder being the z-axis. The two strip cracks inside the medium occupy the region defined by $\theta = 0$, $\theta = \pi$, $-b < r < -a$, $a < r < b$, $-\infty < z < \infty$. The state of stress inside the elastic medium is created by a normal pressure $f(r)$ inside the crack, so that the stress field is the same in the planes normal to the axis of the cylinder. The problem of finding the stress distribution is thus reduced to one of finding the stress distribution in a circular disc of homogeneous isotropic elastic material containing two Griffith cracks occupying the line segments $\theta = 0, \pi$, $-b < r < -a$, $a < r < b$. It may be observed that the four quadrants are symmetric. For a symmetric deformation the displacement vector $\underline{U}$ may be taken to have the components $(u_r(r, \theta), u_\theta(r, \theta), 0)$ and the components of stress tensor will be represented by $\sigma_{rr}(r, \theta)$, $\sigma_{\theta\theta}(r, \theta)$, $\sigma_{r\theta}(r, \theta)$. We shall consider the following problems:

(i) The lateral surface of the cylinder is supposed to be free from stress. The boundary conditions can be written as

$$\sigma_{rr}(\rho, \theta) = \sigma_{r\theta}(\rho, \theta) = 0, \quad 0 \leqslant \theta \leqslant \pi/2. \qquad \text{(2.1)}$$

(ii) The lateral surface of the cylinder is rigidly clamped. The boundary conditions in this case are

$$u_r(\rho, \theta) = u_\theta(\rho, \theta) = 0, \quad 0 \leqslant \theta \leqslant \pi/2. \qquad \text{(2.2)}$$

In addition, on the boundary $\theta = 0$, we have

$$u_\theta(r, 0) = 0, \quad 0 < r < a, \; b < r < \rho \qquad (2.3)$$

$$\sigma_{r\theta}(r, 0) = 0, \quad 0 \leqslant r \leqslant \rho \qquad (2.4)$$

$$\sigma_{\theta\theta}(r, 0) = -2\mu f(r), \quad a < r < b \qquad (2.5)$$

on the boundary $\theta = \pi/2$

$$\sigma_{r\theta}(r, \pi/2) = u_\theta(r, \pi/2) = 0. \qquad (2.6)$$

3. Components of Displacement Vector and Stress Tensor

The appropriate expressions for the components of displacement vector and stress tensor pertaining to the problems under discussion have been derived by Srivastav and Narain (1968). These expressions are

$$u_r(r, \theta) = \int_0^\infty A(\xi)\, e^{-\xi r \sin\theta}[(2-2\eta+\xi r \sin\theta)\sin\theta\cos(\xi r\cos\theta) -(1-2\eta-\xi r\sin\theta)\cos\theta\sin(\xi r\cos\theta)]\,d\xi - \sum_{n=,0}^{\infty}[n\,a_n r^{n-1}+b_n(n+4\eta-2)r^{n+1}]\cos n\theta \qquad (3.1)$$

$$u_\theta(r, \theta) = \int_0^\infty A(\xi)\, e^{-\xi r \sin\theta}[(2-2\eta+\xi r\sin\theta)\cos\theta\cos(\xi r\cos\theta) +(1-2\eta-\xi r\sin\theta)\sin\theta\sin(\xi r\cos\theta)]\,d\xi + \sum_{n=0}^{\infty}[n\,a_n r^{n-1}+b_n(n-4\eta+4)r^{n+1}]\sin n\theta \qquad (3.2)$$

$$\frac{\sigma_{rr}(r, \theta)}{2\mu} = -\int_0^\infty \xi A(\xi)\, e^{-\xi r\sin\theta}[(1-\xi r\sin\theta\cos 2\theta)\cos(\xi r\cos\theta) +\xi r\sin\theta\sin 2\theta\sin(\xi r\cos\theta)]\,d\xi - \sum_{n=0}^{\infty}[n(n-1)a_n r^{n-2}+(n+1)(n-2)b_n r^n]\cos n\theta \qquad (3.3)$$

$$\frac{\sigma_{\theta\theta}(r, \theta)}{2\mu} = -\int_0^\infty \xi A(\xi)\, e^{-\xi r\sin\theta}[(1+\xi r\sin\theta\cos 2\theta)\cos(\xi r\cos\theta) -\xi r\sin\theta\sin 2\theta\sin(\xi r\cos\theta)]\,d\xi + \sum_{n=0}^{\infty}[n(n-1)a_n r^{n-2}+(n+1)(n-2)b_n r^n]\cos n\theta \qquad (3.4)$$

$$\frac{\sigma_{r\theta}(r, \theta)}{2\mu} = -\int_0^\infty \xi^2 r A(\xi)\, e^{-\xi r\sin\theta}\sin\theta\sin(2\theta+\xi r\cos\theta)\,d\xi + \sum_{n=0}^{\infty}[n(n-1)a_n r^{n-2}+n(n+1)b_n r^n]\sin n\theta. \qquad (3.5)$$

The expressions for displacements u_r and u_θ derived by Srivastav and Narain contain a misprint. In formulae (3.2.5) and (3.2.6) $1/(1+\eta)$ should be replaced by $(1-\eta)$.

4. Some Useful Results

We list here, for ready reference, some integrals that we shall use in the future sections. All these results are well known and can be found in Gradshteyn and Ryzhik (1965).

$$\int_a^b \{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\, dt = (1/b)F\left(\frac{\pi}{2}, q\right) = F/b \quad \ldots \quad (4.1)$$

$$\int_a^b \{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\, t^2\, dt = bE\left(\frac{\pi}{2}, q\right) = bE \quad \ldots \quad (4.2)$$

where $q = (1-a^2/b^2)^{\frac{1}{2}}$ and F and E are complete elliptic integrals of the first and second kind. It is simple to derive the following integrals:

$$\int_a^b \frac{t\, dt}{\{(t^2-a^2)(b^2-t^2)\}^{\frac{1}{2}}(t^2-x^2)} = \begin{Bmatrix} \frac{\pi}{2}\{(a^2-x^2)(b^2-x^2)\}^{\frac{1}{2}} & , \ 0<x<a \\ 0 & , \ a<x<b \\ -\frac{\pi}{2}\{(x^2-a^2)(x^2-b^2)\}^{\frac{1}{2}}, & x>b \end{Bmatrix} \quad \ldots \quad (4.3)$$

$$\int_0^\infty \left\{\frac{e^{-\xi\rho\sin\theta}\sin\xi x}{\xi}\right\} \begin{matrix}\cos(\xi\rho\cos\theta) \\ \sin(\xi\rho\cos\theta)\end{matrix}\, d\xi = \sum_{n=0}^{\infty} \frac{1}{(2n+1)} (x/\rho)^{2n+1} \begin{matrix}\sin(2n+1)\theta \\ \cos(2n+1)\theta\end{matrix}$$

$$\int_0^\infty \left\{e^{-\xi\rho\sin\theta}\sin\xi x\right\} \begin{matrix}\cos(\xi\rho\cos\theta) \\ \sin(\xi\rho\cos\theta)\end{matrix}\, d\xi = \mp \sum_{n=0}^{\infty} \frac{x^{2n+1}}{\rho^{2n+2}} \begin{matrix}\cos(2n+2)\theta \\ \sin(2n+2)\theta\end{matrix}$$

$$\int_0^\infty \left\{\xi e^{-\xi\rho\sin\theta}\sin\xi x\right\} \begin{matrix}\cos(\xi\rho\cos\theta) \\ \sin(\xi\rho\cos\theta)\end{matrix}\, d\xi = -\sum_{n=0}^{\infty} (2n+2)\frac{x^{2n+1}}{\rho^{2n+3}} \begin{matrix}\sin(2n+3)\theta \\ \cos(2n+3)\theta.\end{matrix}$$

With the help of these integrals we can easily show that

$$\int_0^\infty e^{-\xi\rho\sin\theta}[(2-2\eta+\xi\rho\sin\theta)\sin\theta\cos(\xi\rho\cos\theta)$$

$$-(1-2\eta-\xi\rho\sin\theta)\cos\theta\sin(\xi\rho\cos\theta)]\,\frac{\sin\xi x}{\xi}\,d\xi$$

$$= 1+\sum_{n=1}^{\infty}\left\{\frac{n+1}{2n+1}(x/\rho)^{2n+1} - \frac{n+1-2\eta}{2n-1}(x/\rho)^{2n-1}\right\}\cos 2n\theta \quad \ldots \quad (4.4)$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}[(2-2\eta+\xi\rho\sin\theta)\cos\theta\cos(\xi\rho\cos\theta)$$

$$+(1-2\eta-\xi\rho\sin\theta)\sin\theta\sin(\xi\rho\cos\theta)]\,\frac{\sin\xi x}{\xi}\,d\xi$$

$$= \sum_{n=0}^{\infty}\left[\frac{n+1}{2n+1}(x/\rho)^{2n+1} - \frac{n+2\eta-2}{2n-1}(x/\rho)^{2n-1}\right]\sin 2n\theta \quad \ldots \quad (4.5)$$

$$-\int_0^\infty \xi\rho\, e^{-\xi\rho \sin\theta} \sin\theta \sin(2\theta+\xi r\cos\theta)\sin \xi x\, d\xi$$

$$= \sum_{n=0}^{\infty}\left[n\,\frac{x^{2n-1}}{\rho^{2n}}-(n+1)\,\frac{x^{2n+1}}{\rho^{2n+2}}\right]\sin 2n\theta \quad \ldots \quad (4.6)$$

$$-\int_0^\infty e^{-\xi\rho\sin\theta}[(1-\xi\rho\sin\theta\cos 2\theta)\cos(\xi r\cos\theta)$$

$$+\xi\rho\sin 2\theta\sin\theta\sin(\xi\rho\cos\theta)]\sin\xi x\, d\xi$$

$$= -x/\rho^2+\sum_{n=1}^{\infty}(n+1)\left[\frac{x^{2n-1}}{\rho^{2n}}-\frac{x^{2n+1}}{\rho^{2n+2}}\right]\cos 2n\theta. \quad \ldots \quad (4.7)$$

The solution of integral equation

$$\frac{2}{\pi}\int_a^b \frac{x\,h(x^2)}{x^2-y^2}\,dx = p(y), \quad y\in(a,b) \quad \ldots \quad (4.8)$$

as given in Srivastava and Lowengrub (1970) is

$$h(x^2) = -\mathfrak{H}(p(y))+c/\{(x^2-a^2)(b^2-x^2)\}^{\frac{1}{2}} \quad \ldots \quad (4.9)$$

where $\mathfrak{H}(.)$ is the Hilbert operator, and

$$\mathfrak{H}(p(y)) = \frac{2}{\pi}\int_a^b \left\{\frac{(x^2-a^2)(b^2-y^2)}{(b^2-x^2)(y^2-a^2)}\right\}^{\frac{1}{2}} p(y)\,\frac{y\,dy}{y^2-x^2}. \quad \ldots \quad (4.10)$$

5. Conditions on Cracks Face

We divide the solution of the problem into two parts. In the first part we consider the relations which exist between certain arbitrary functions and the unknown coefficients a_n and b_n, if the boundary conditions on the line $\theta=0$ and $\theta=\pi/2$ are to be satisfied. The boundary conditions (2.6) on $\theta=\pi/2$ are satisfied if and only if the coefficients a_{2n+1} and b_{2n+1} are all zero. With this choice of a_{2n+1} and b_{2n+1}, we see that (2.4) is also satisfied. The conditions (2.3) and (2.5) lead to the triple integral equations

$$\begin{cases} \int_0^\infty A(\xi)\cos\xi r\, d\xi = 0, \quad 0<r<a, \\ \int_0^\infty \xi\, A(\xi)\cos r\, d\xi = f(r)+\sum_{n=0}^{\infty}[a_{2n}2n(2n-1)r^{2n-2}+b_{2n}(2n+1)(2n+2)r^{2n}] \\ \qquad = \frac{\pi}{2}F(r), \quad a<r<b \\ \int_0^\infty A(\xi)\cos\xi r\, d\xi = 0, \quad r>b \end{cases} \quad \ldots \quad (5.1)$$

where

$$F(r) = \frac{2}{\pi}\left[f(r)+\sum_{n=0}^{\infty}(2n+2)(2n+1)(a_{2n+2}+b_{2n})r^{2n}\right].$$

The solution of the above set of triple integral equations as given by Srivastava and Lowengrub (1970) is given by

$$A(\xi) = \frac{1}{\xi}\int_a^b h(t^2) \sin \xi t \, dt \qquad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

where $h(t^2)$ is determined by

$$h(t^2) = -\frac{2}{\pi} \mathfrak{H}(F(r)) + c/\{(t^2-a^2)(b^2-t^2)\}^{\frac{1}{2}} \qquad \ldots \quad \ldots \quad (5.3)$$

satisfying the condition $\int_b^a h(t^2) \, dt = 0$ and c is an arbitrary constant. Thus (5.3) gives the relation connecting $h(t^2)$ with a_{2n} and b_{2n}.

6. Conditions on the Free Boundary

We now complete the solution by satisfying the boundary conditions (2.1) and (2.2) for the two problems.

Problem 1

The conditions $\sigma_{rr}(\rho, \theta) = \sigma_{r\theta}(\rho, \theta) = 0$ for $0 \leqslant \theta \leqslant \pi/2$ are satisfied provided

$$\int_0^\infty \xi A(\xi) \, e^{-\xi\rho \sin\theta} [(1-\xi\rho \sin\theta \cos 2\theta) \cos (\xi\rho \cos\theta) + \xi\rho \sin\theta \sin 2\theta \sin (\xi\rho \cos\theta)] \, d\xi$$

$$+\sum_{n=0}^{\infty} [2n(2n-1)a_{2n} \rho^{2n-2} + (2n+1)(2n-2)b_{2n} \rho^{2n}] \cos 2n\theta = 0 \qquad \ldots \quad \ldots \quad (6.1)$$

$$-\int_0^\infty \xi^2\rho \, A(\xi) \, e^{-\xi\rho \sin\theta} \sin\theta \sin (2\theta + \xi\rho \cos\theta) \, d\xi$$

$$+\sum_{n=0}^{\infty} [2n(2n-1)a_{2n} \rho^{2n-2} + 2n(2n+1)b_{2n} \rho^{2n}] \sin 2n\theta = 0. \qquad \ldots \quad \ldots \quad (6.2)$$

Substituting the value of $A(\xi)$ from (5.2) in the above equations, inverting the order of integration and after using (4.6) and (4.7) we get

$$-\frac{1}{\rho^2}\int_a^b x \, h(x^2) \, dx + 2b_0 + \sum_{n=1}^{\infty}\Bigg[(n+1)\int_a^b h(x^2)\left\{\frac{x^{2n-1}}{\rho^{2n}} - \frac{x^{2n+1}}{\rho^{2n+2}}\right\} dx$$

$$-\{2n(2n-1)a_{2n} \rho^{2n-2} + (2n+1)(2n-2)b_{2n} \rho^{2n}\} \cos 2n\theta\Bigg] = 0 \quad \ldots \quad (6.3)$$

$$\sum_{n=0}^{\infty}\Bigg[\int_a^b h(x^2)\left\{n\frac{x^{2n-1}}{\rho^{2n}} - (n+1)\frac{x^{2n+1}}{\rho^{2n+2}}\right\} dx$$

$$+\{2n(2n-1)a_{2n} \rho^{2n-2} + 2n(2n+1)b_{2n} \rho^{2n}\} \sin 2n\theta\Bigg] = 0 \quad \ldots \quad \ldots \quad (6.4)$$

for $0 \leqslant \theta \leqslant \pi/2$. From these equations we derive

$$2b_0 = \frac{1}{\rho^2}\int_a^b x\, h(x^2)\, dx$$

$$2n(2n-1)a_{2n}\,\rho^{2n-2} = \int_a^b \left\{2n^2 \frac{x^{2n-1}}{\rho^{2n}} - (2n-1)(n+1)\frac{x^{2n+1}}{\rho^{2n+2}}\right\} h(x^2)\, dx$$

$$2(2n+1)b_{2n}\,\rho^{2n} = \int_a^b \left\{(2n+2)\frac{x^{2n+1}}{\rho^{2n+2}} - (2n+1)\frac{x^{2n-1}}{\rho^{2n}}\right\} h(x^2)\, dx$$

for $n \geqslant 1$.
With the help of the above values of a_{2n} and b_{2n} we derive

$$\sum_{n=0}^{\infty} (2n+1)(2n+2)\{a_{2n+2}+b_{2n}\}r^{2n} = \int_a^b x\, h(x^2)\, N(x, r)\, dx$$

where

$$N(x, r) = \frac{3}{\rho^2} - \frac{2x^2}{\rho^4} - \frac{1}{x^2}\sum_{n=1}^{\infty}\left(\frac{xr}{\rho^2}\right)^{2n}\{(n+1)(2n+1) - 4(n+1)^2(x/\rho)^2$$
$$+(n+2)(2n+1)(x/\rho)^4\}.$$

It is interesting to note that the above derivations are independent of η. Hence, in the case when the lateral surface of the cylinder is stress-free, the distribution of stress in the vicinity of crack is independent of the mechanical properties of the cylinder.

Problem 2

When the curved surface of cylinder is rigidly clamped, the boundary conditions to be satisfied are

$$u_r(\rho, \theta) = u_\theta(\rho, \theta) = 0 \quad \text{for } 0 \leqslant \theta \leqslant \pi/2.$$

In contrast to the first problem in this case the coefficients a_{2n} and b_{2n} are dependent on η. Hence, the elastic properties of the cylinder have to be accounted for. With the object of simplifying the calculations we assume that $\eta = \frac{1}{4}$. An exactly similar procedure as for the first problem and with the help of results (4.4) and (4.5), one can calculate the coefficients.

For $n \geqslant 1$

$$4b_{2n}\,\rho^{2n+1} = \int_a^b h(x^2)\left\{-\frac{2n+2}{2n+1}(x/\rho)^{2n+1} + (x/\rho)^{2n-1}\right\} dx$$

$$4na_{2n}\,\rho^{2n-1} = \int_a^b h(x^2)\left\{-\frac{2n^2+3}{4n-2}(x/\rho)^{2n-1} + (n+1)(x/\rho)^{2n+1}\right\} dx$$

$$b_0 = 0.$$

With the help of these values we can show that

$$\sum_{n=0}^{\infty} (2n+1)(2n+2)\{a_{2n+2}+b_{2n}\}r^{2n} = \int_a^b N(x, r)\, h(x^2)\, x\, dx$$

where

$$N(x,\,r) = -\frac{5}{4\rho^2}+\frac{x^2}{\rho^4}+\frac{1}{x^2}\sum_{n=1}^{\infty}\left(\frac{xr}{\rho^2}\right)^{2n}\{(n+1)(n+\tfrac{1}{2})-(n+\tfrac{1}{2})(n+2)(x/\rho)^2$$
$$+\tfrac{3}{4}(2n^2+4n+3)(x/\rho)^4\}\,.$$

Hence

$$F(r) = \frac{2}{\pi}\left[f(r)+\int_a^b x\,h(x^2)\,N(x,\,r)\,dx\right]. \qquad \ldots \qquad \ldots \quad (6.5)$$

Substituting this value of $F(r)$ in (5.3), we get

$$h(t^2) = -\frac{2}{\pi}\mathfrak{H}\left[f(r)+\int_a^b x\,h(x^2)\,N(x,\,r)\,dx\right]+c/\{(b^2-t^2)(t^2-a^2)\}^{\frac{1}{2}}. \qquad (6.6)$$

When $\rho \gg b$, $N(x,\,r)$ can be written as

$$N(x,\,r) = \frac{A_0}{\rho^2}+(1/\rho^4)(A_1x^2+A_2r^2)+\frac{A_3}{\rho^6}x^2r^2+O(\rho^{-8}), \qquad \ldots \quad (6.7)$$

where the constants A's have the following values for the two problems:

	Problem 1	*Problem 2*
A_0	3	$-5/4$
A_1	-2	1
A_2	-6	3
A_3	16	$-9/2$

Substituting the value of $N(x,\,r)$ in (6.6) and, after some calculations, we get

$$h(t^2) = -\frac{2}{\pi}\mathfrak{H}(f(r))+\frac{2}{\pi}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}}\int_a^b x\,h(x^2)\left[\left(\frac{A_0}{\rho^2}+\frac{A_1x^2}{\rho^4}\right)\right.$$
$$\left.+\left(\frac{A_2}{\rho^4}+\frac{A_3x^2}{\rho^6}\right)\left(t^2+\frac{a^2-b^2}{2}\right)+O(\rho^{-8})\right]dx+\frac{c}{T} \qquad \ldots \quad \ldots \quad (6.8)$$

where $T = \{(t^2-a^2)(b^2-t^2)\}^{\frac{1}{2}}$.

Utilization of condition $\int_a^b h(t^2)\,dt = 0$ gives

$$0 = -\frac{2}{\pi}\int_a^b \mathfrak{H}(f(r))\,dt+\frac{2}{\pi}\int_a^b x\,h(x^2)\left[b\left(\frac{A_0}{\rho^2}+\frac{A_1x^2}{\rho^4}\right)\left(E-\frac{a^2}{b^2}F\right)\right.$$
$$\left.+\frac{b}{6}\left(\frac{A_2}{\rho^4}+\frac{A_3x^2}{\rho^6}\right)\left\{(a^2+b^2)E-\left(\frac{a^2}{b^2}\right)(3a^2-b^2)F\right\}+O(\rho^{-8})\right]dx+cF/b. \qquad (6.9)$$

Eliminating c between (6.8) and (6.9) we get

$$h(t^2) = \phi(t^2)+\int_a^b x\,h(x^2)\,M(x,\,t^2)\,dx \quad \ldots \qquad \ldots \qquad \ldots \quad (6.10)$$

where

$$M(x,\, t^2) = \frac{2}{\pi T}\left[(t^2-b^2E/F)\left(\frac{A_0}{\rho^2}+\frac{A_1x^2}{\rho^4}\right)+\frac{1}{2}\left(\frac{A_2}{\rho^4}+\frac{A_3x^2}{\rho^6}\right)\{2t^4-(a^2+b^2)t^2+\beta\}+O(\rho^{-8})\right]$$

$$\beta = \{2a^2/3-(a^2+b^2)E/3F\}b^2$$

and

$$\phi(t^2) = -\frac{2}{\pi}\mathfrak{H}(f(r))+\frac{2b}{\pi FT}\int_a^b \mathfrak{H}(f(r))\, dt.$$

Since $\ldots \rho \gg b$, $|M(x,\, t^2)| < K$, where $K < 1$, the solution of (6.10) can be written as

$$h(t^2) = h_0(t^2)+h_1(t^2)/\rho+h_2(t^2)/\rho^2+\ldots \qquad \ldots \quad (6.11)$$

where

$$h_0(t^2) = \phi(t^2), \quad h_1(t^2) = h_3(t^2) = h_5(t^2) = \ldots = 0,$$

$$h_2(t^2) = \frac{2A_0(t^2-b^2E/F)}{\pi T}\int_a^b x\, h_0(x^2)\, dx,$$

$$h_4(t^2) = \frac{2}{\pi T}\left[(t^2-b^2E/F)\int_a^b \{A_1x^2h_0(x^2)+A_0h_2(x^2)\}x\, dx + \frac{A_2}{2}\{2t^4-(a^2+b^2)t^2+\beta\}\int_a^b x\, h_0(x^2)\, dx\right],$$

$$h_6(t^2) = \frac{2}{\pi T}\left[(t^2-b^2E/F)\int_a^b x\{A_0h_4(x^2)+A_1x^2h_2(x^2)\}dx + \tfrac{1}{2}\{2t^4-(a^2+b^2)t^2+\beta\}\int_a^b x(A_3x^2h_0(x^2)+A_2h_2(x^2))\, dx\right].$$

We now consider the physically important case when the cracks are opened by the application of constant pressure p_0, so that $f(r) = p_0/2\mu = P$ (say). So that

$$h_0(t^2) = \frac{2P}{\pi T}(t^2-b^2E/F)$$

$$h_2(t^2) = \frac{2P}{\pi T}c_0(t^2-b^2E/F)$$

$$h_4(t^2) = \frac{2P}{\pi T}[c_1(t^2-b^2E/F)+c_2\{2t^4-(a^2+b^2)t^2+\beta\}]$$

$$h_6(t^2) = \frac{2P}{\pi T}[c_3(t^2-b^2E/F)+c_4\{2t^4-(a^2+b^2)t^2+\beta\}]$$

where

$$c_0 = A_0\left\{\frac{a^2+b^2}{2} - b^2\frac{E}{F}\right\}$$

$$c_1 = c_0^2 + A_1\left\{\frac{c_0(a^2+b^2)}{A_0} + \left(\frac{a^2-b^2}{2}\right)^2\right\}\Big/2$$

$$c_2 = \frac{A_2 c_0}{2A_0}$$

$$c_3 = c_0\left\{c_1 + (c_1 - c_0^2)\left(1+\frac{A_2}{3A_1}\right)\right\}$$

$$c_4 = 2c_0c_2 + A_3(c_1-c_0^2)/A_1.$$

The calculations for the higher powers of ρ^{-1} can be easily obtained by continuing the iterative procedure.

7. The Stress Intensity Factors

The stress intensity factors at the two ends of the crack are given by

$$N_a = \lim_{r\to a^-} (a-r)^{\frac{1}{2}}[\sigma_{\theta\theta}(r, 0)] \quad 0 < r < a \qquad .. \quad .. \quad (7.1)$$

$$N_b = \lim_{r\to b^+} (r-b)^{\frac{1}{2}}[\sigma_{\theta\theta}(r, 0)] \quad r > b. \qquad .. \quad .. \quad (7.2)$$

From (3.4) we observe that

$$\frac{\sigma_{\theta\theta}(r, 0)}{2\mu} = -\frac{d}{dr}\int_0^\infty A(\xi)\sin\xi r\, d\xi + \sum_{n=0}^\infty (2n+1)(2n+2)\{a_{2n+2}+b_{2n}\}r^{2n}.$$

Substituting the value of $A(\xi)$ from (5.2) and the values of a_{2n} and b_{2n} from section 6, we get

$$\frac{\sigma_{\theta\theta}(r, 0)}{2\mu} = -\int_a^b \frac{x\,h(x^2)}{x^2-r^2}\,dx + \int_a^b x\,h(x^2)\,N(r, t)\,dx \qquad .. \quad (7.3)$$

where $N(r, t)$ has the value given in (6.7).

For $0 < r < a$, we have

$$-\int_a^b \frac{x\,h(x^2)}{x^2-r^2}\,dx = \frac{p_0}{2\mu}\Bigg[\left(\frac{Eb^2/F-r^2}{R}-1\right)(1+c_0/\rho^2+c_1/\rho^4+c_3/\rho^6)$$
$$-\left(2r^2+\frac{2r^4-(a^2+b^2)r^2+\beta}{R}\right)(c_2/\rho^4+c_4/\rho^6)+O(\rho^{-8})\Bigg]$$

where

$R = \{(a^2-r^2)(b^2-r^2)\}^{\frac{1}{2}}$ and for $r > b$, we have

$$-\int_a^b \frac{x\,h(x^2)}{x^2-r^2}\,dx = \frac{p_0}{2\mu}\Bigg[\left(\frac{r^2-b^2E/F}{R_1}-1\right)(1+c_0/\rho^2+c_1/\rho^4+c_3/\rho^6)$$
$$+\left(\frac{2r^4-(a^2+b^2)r^2+\beta}{R_1}-2r^2\right)(c_2/\rho^4+c_4/\rho^6)+O(\rho^{-8})\Bigg],$$

where

$$R_1 = \{(r^2-a^2)(r^2-b^2)\}^{\frac{1}{2}}.$$

It is easy to show that

$$\int_a^b x\, h(x^2)\, N(x, r)\, dx = c_0/\rho^2 + \frac{1}{\rho^4}\left\{c_0^2 + c_0(c_1-c_0^2) + \frac{A_1 c_0 r^2}{A_0}\right\}$$
$$+\frac{1}{\rho^6}\left\{c_0c_1 + c_0(c_1-c_0^2)\left(1+\frac{2A_0}{3A_1}\right)\right.$$
$$\left. + r^2\left(\frac{A_1c_0^2}{A_0} + \frac{A_3(c_1-c_0^2)}{A_1}\right)\right\} + O(\rho^{-8}).$$

In the above calculations we have taken the help of the first three results given in section 4. Hence the stress intensity factors as estimated from (7.1) and (7.2) are

$$N_a = \frac{p_0}{\{2a(b^2-a^2)\}^{\frac{1}{2}}}[(Eb^2/F-a^2)(1+c_0/\rho^2+c_1/\rho^4+c_3/\rho^6)$$
$$+(c_2/\rho^4+c_4/\rho^6)(a^2(b^2-a^2)-\beta)+O(\rho^{-8})] \quad \ldots \quad \ldots \quad \ldots \quad (7.4)$$

$$N_b = \frac{p_0}{\{2b(b^2-a^2)\}^{\frac{1}{2}}}[b^2(1-E/F)(1+c_0/\rho^2+c_1/\rho^4+c_3/\rho^6)$$
$$+(c_2/\rho^4+c_4/\rho^6)(b^2(b^2-a^2)+\beta)+O(\rho^{-8})]. \quad \ldots \quad \ldots \quad (7.5)$$

8. The Shape of the Deformed Cracks and the Crack Energy

The displacement along the crack is given by

$$u_\theta(r, 0) = 2(1-\eta)\int_0^\infty A(\xi)\cos\xi r\, d\xi, \quad a < r < b. \quad \ldots \quad (8.1)$$

Substituting for $A(\xi)$ and performing the interchange of the order of integration, we obtain

$$u_\theta(r, 0) = \pi(1-\eta)\int_r^b h(t^2)\, dt, \quad a < r < b. \quad \ldots \quad \ldots \quad (8.2)$$

Hence

$$\frac{u_\theta(r, 0)}{2P(1-\eta)} = \left[b\left\{1+\frac{c_0}{\rho^2}+\frac{1}{\rho^4}\left(c_1+\frac{(a^2+b^2)c_2}{3}\right)+\frac{1}{\rho^6}\left(c_3+\frac{(a^2+b^2)c_4}{3}\right)\right\}\right.$$
$$\left.\times\left\{E(\phi, q)-\frac{E}{F}F(\phi, q)\right\}+\left(\frac{c_2}{\rho^4}+\frac{c_4}{\rho^6}\right)\left\{(r^2-a^2)(b^2-r^2)\right\}^{\frac{1}{2}}\frac{2r}{3}+O(\rho^{-8})\right]$$
$$\ldots \quad (8.3)$$

where

$$\sin\phi = \{(b^2-r^2)/(b^2-a^2)\}^{\frac{1}{2}}.$$

The total energy required to open two cracks is

$$W = 2\int_a^b \sigma_{\theta\theta}(r, 0)\, u_\theta(r, 0)\, dr = 2\pi(1-\eta)p_0\int_a^b t\, h(t^2)\, dt.$$

Since $\sigma_{\theta\theta}(r, 0) = p_0$, a constant, and $\int_a^b h(t^2)\,dt = 0$, hence we have

$$W = \frac{2\pi(1-\eta^2)p_0^2}{Y}\left[\frac{c_0}{A_0}\left(1+\frac{c_0}{\rho^2}+\frac{c_1}{\rho^4}+\frac{c_3}{\rho^6}\right)+\left(\frac{c_2}{\rho^4}+\frac{c_4}{\rho^6}\right)\left(\left(\frac{b^2-a^2}{2}\right)^2+\beta\right)+O(\rho^{-8})\right]. \quad .. \quad (8.4)$$

The case $a = 0$, that is when the two cracks merge into a single crack of length $2b$, yields $E \to b$, $F \to \infty$, $E/F \to 0$. After some calculations one finds that the crack energy for the two problems are:

Problem 1

$$W = \frac{\pi(1-\eta^2)b^2p_0^2}{Y}[1+1{\cdot}5x^2+0{\cdot}75x^4+0{\cdot}375x^6+O(x^8)] \quad .. \quad (8.5)$$

Problem 2

$$W = \frac{\pi(1-\eta^2)b^2p_0^2}{Y}[1-0{\cdot}625x^2+0{\cdot}95312x^4-1{\cdot}90936x^6+O(x^8)] \quad .. \quad (8.6)$$

where $x = b/\rho$.

The Griffith's criterion that the crack may spread is

$$\frac{\partial}{\partial b}(W-U) = 0$$

where U, the surface energy of the crack, is given in terms of the surface tension T_0 of the material, by

$$U = 2T_0 b.$$

This leads to the expressions for the critical value of the applied pressure to cause the crack to spread when the length is $2b$. The values of critical pressure p_{cr} are:

Problem 1

$$p_{cr} = p_{cr}^{\infty}[1-1{\cdot}5x^2+1{\cdot}6875x^4-1{\cdot}59375x^6+O(x^8)] \quad .. \quad (8.7)$$

Problem 2

$$p_{cr} = p_{cr}^{\infty}[1+0{\cdot}625x^2-0{\cdot}54625x^4+0{\cdot}69546x^6+O(x^8)] \quad .. \quad (8.8)$$

where $p_{cr}^{\infty} = \left[\frac{2YT_0}{\pi b(1-\eta^2)}\right]^{\frac{1}{2}}$ is the critical pressure for an infinite medium containing a crack.

The solution of the problem of finding the distribution of stress in the cylinder for the case when the cracks inside the medium occupy the region defined by $\theta = 0$, $\theta = \pi$, $a < |r| < \rho$, $-\infty < z < \infty$ can be obtained as the limiting case if we suppose that $b \to \rho$.

Addendum

The solution of Fredholm integral eqn. (6.10) in the form (6.11) needs a justification which is as follows: We have called (6.10) a Fredholm integral equation although its kernel $M(x, t^2)$ does not belong to the type of kernel which as a rule is called regular. This kernel is not $L^2(a, b)$ as it has weak singularities at the end points. In spite of this, all fundamental Fredholm theorems are applicable to eqn. (6.10), if they are stated in a suitable manner. This will be shown by reduction of (6.10) to a Fredholm integral equation with bounded $L^2(a, b)$ kernel. Writing (6.10) as

$$H(t^2)-\int_a^b x\,H(x^2)\frac{R(x, t^2)}{T(x^2)}\,dx = T(t^2)\,\phi(t^2) \quad \text{(i)}$$

where $H(t^2) = T(t^2)\,h(t^2)$, $R(x, t^2) = T(t^2)\,M(x, t^2)$.

It is easy to show that $R(x, t^2)$ is bounded with respect to variable t. The right-hand side of (i) is easily seen to be bounded. The variable x will now be replaced by θ, by writing $x^2 = a^2\sin^2\theta+b^2\cos^2\theta$. With this transformation (i) reduces to

$$H(t^2)-\int_0^{\frac{\pi}{2}} x^2 H(x^2)\,R(x, t^2)\,d\theta = T(t^2)\,\phi(t^2) \quad \text{(ii)}$$

This is a Fredholm integral equation with a bounded $L^2(0, \pi/2)$ kernel. Thus the solution of (6.10) is equivalent to the solution of the Fredholm integral equation (ii) in ordinary sense. Consequently we can conclude that the Fredholm theorems are valid for (6.10).

The expression (8.3) for normal displacement of cracks can be written as

$$\frac{Yu_\theta(r, 0)}{2p_0(1-\eta^2)} = \{(1+c_0\rho^{-2}+c_1\rho^{-4}+c_3\rho^{-6})+\tfrac{1}{3}(a^2+1)(c_2\rho^{-4}+c_4\rho^{-6})\}\left\{E(\phi, q)-\frac{E}{F}F(\phi, q)\right\}$$
$$+\frac{2r}{3}\left(\frac{c_2}{\rho^4}+\frac{c_4}{\rho^6}\right)\{(r^2-a^2)(1-r^2)\}^{\frac{1}{2}}+O(\rho^{-8}). \quad \text{(iii)}$$

The expression for distribution of stress at origin is found from (7.3) by putting $r = 0$ and it is

$$\frac{\sigma_{\theta\theta}(0, 0)}{p_0} = (1+c_0\rho^{-2}+c_1\rho^{-4}+c_3\rho^{-6})(E/aF-1)-(c_2\rho^{-4}+c_4\rho^{-6})(\tfrac{2}{3}a-\tfrac{1}{3}(a^2+1)E/aF)$$
$$+c_0\rho^{-2}+\{c_0^2+c_0(c_1-c_0^2)\}\rho^{-4}+c_0c_1\rho^{-6}+O(\rho^{-8}). \quad \text{(iv)}$$

The numerical calculations were carried out for radius of the cylinder, $\rho = 2$ units of length, and for $a = 0{\cdot}500$, $0{\cdot}259$, $0{\cdot}0175$. The curves for normal displacement of cracks are shown in Fig. 1. The continuous curves correspond to the problem under discussion and the dotted curves to the problem considered by Tranter (1961). It is obvious that the opening of cracks is more as compared to the problem considered by Tranter.

Figure 2 shows variation of stress at origin with $a = 0{\cdot}2$, $0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$, $1{\cdot}0$. The continuous curve corresponds to the problem under consideration and the dotted one to that of Tranter's problem. The concentration of stress at origin is more in this case.

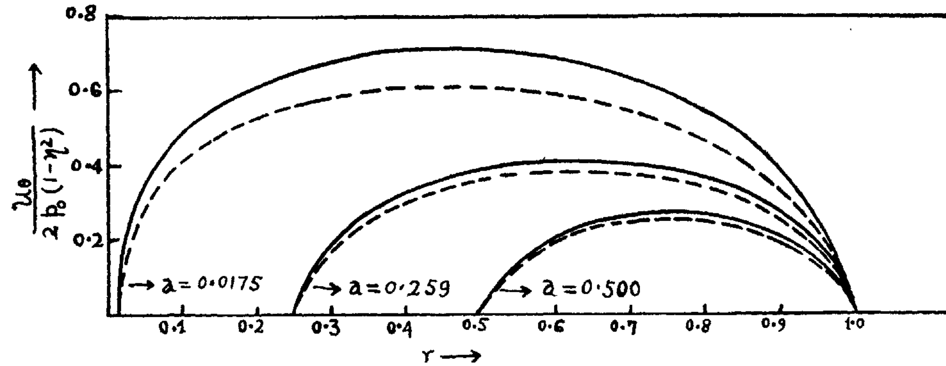

FIG. 1. Variation of normal displacement with a and r.

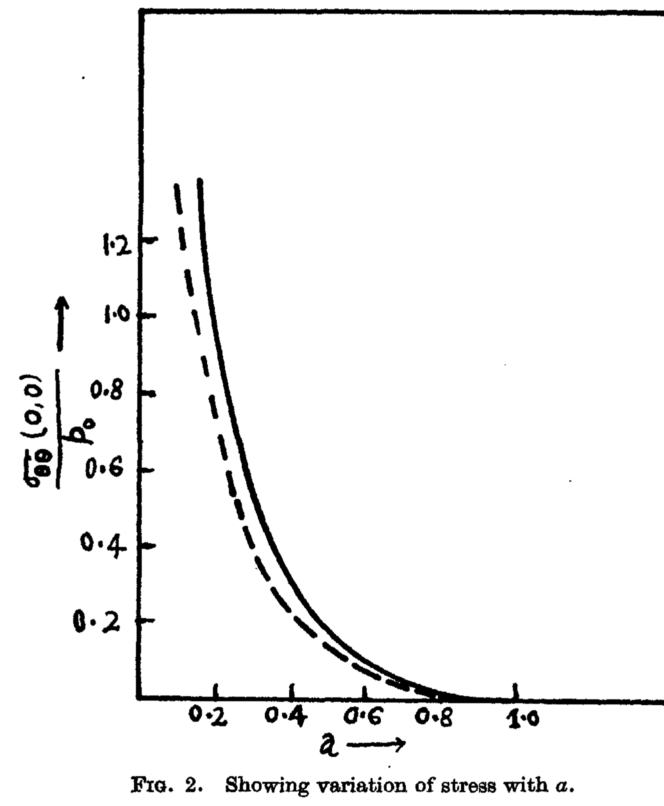

FIG. 2. Showing variation of stress with a.

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