INFINITESIMAL GEOMETRY IN HILBERT SPACE—I

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The concept of function space is utilized in building up this infinitesimal geometry in Hilbert space. In an infinite-dimensional Euclidean space characterized by an infinity of mutually orthogonal coordinate axes all geometrical entities are defined within an infinitesimal domain in the neighbourhood of the origin. The analytical process associated with this development is, however, an extension of that adopted by the author (Ghosh 1936*b*) in the treatment of the geometry of hyperspace.

§ 1. Consider a finite closed interval $[a, b]$ in a one-dimensional space. Let the interval be divided into sub-intervals by the points

$$a = x_0 < x_1 < x_2 \ldots < x_n = b.$$

Denote each sub-interval $x_q - x_{q-1}$ by dx_q. Then obviously the sum

$$\sum_{q=1}^{n} dx_q$$

has the constant value $(b-a)$ independent of the mode of subdivision. When the number n of these sub-intervals approaches infinity each of the sub-intervals is regarded as an infinitesimal element of the infinitesimal geometry.

To represent them along the infinite number of mutually orthogonal coordinate axes in a real Hilbert space we introduce the notion of infinitesimal coordinate vectors with norm $\sqrt{dx_q}$ so that the coordinate axes are successively labelled by $\sqrt{dx_1}, \sqrt{dx_2} \ldots \sqrt{dx_n}$, $n \to \infty$, thus forming the infinitesimal basic vector $\sqrt{dx}$, the square of the norm of the vector $\sqrt{dx}$ being given by

$$\int_a^b dx = b-a. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

We consider now a finite real-valued function $f(x)$ defined on the closed interval $[a, b]$ and form an infinitesimal vector $f(x)\sqrt{dx}$ having components along the coordinate axes $\sqrt{dx_q}$ given by $f(x_q)\sqrt{dx_q}$. Following the process of Riemann integration of a function between limits (a, b) we define the square of the norm of the vector $f(x)\sqrt{dx}$ as

$$\int_a^b \{f(x)\}^2\, dx \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

and the scalar product of this vector with the basic vector $\sqrt{dx}$ as

$$\int_a^b f(x)\,dx. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

§ 2. Let $\alpha^i = f^i(x_q)\sqrt{dx_q}$ $(i = 1, 2, \ldots m)$ represent m vectors linearly independent, then the scalar determinant (Ghosh 1936*a*)

$$\Delta_{\alpha^m} = \begin{Bmatrix} \alpha^1 & \alpha^2 & \ldots & \alpha^m \\ \alpha^1 & \alpha^2 & \ldots & \alpha^m \end{Bmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

of mth order having the $(i-j)$th element the scalar product of α^i and α^j chosen from the first and the second rows respectively. Denoting the scalar product by the symbol $\begin{Bmatrix} \alpha^i \\ \alpha^j \end{Bmatrix}$ we have

$$\begin{Bmatrix} \alpha^i \\ \alpha^j \end{Bmatrix} = \int_a^b f^i(x)\,f^j(x)\,dx. \qquad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

The scalar determinant (2.1) gives the square of the content of the m-dimensional infinitesimal parallelotope bounded by the infinitesimal vectors as conterminous edges. In ordinary space such determinants are called Gram's determinants.

The equation of an m-dimensional linear subspace L_m from a point A in the infinitesimal space is of the form

$$\rho = A + \sum_{i=1}^{m} s_i \alpha^i \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

where A is the vector from the origin to the point A and s_i are variable parameters. The determinant Δ_{α^m} will be called the dominant of the linear subspace L_m.

The equation (2.3) shows for all ρ-points lying on the linear subspace L_m the scalar determinant

$$\begin{Bmatrix} \rho-A\ \alpha^1 & \alpha^2 & \ldots & \alpha^m \\ \rho-A\ \alpha^1 & \alpha^2 & \ldots & \alpha^m \end{Bmatrix} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

To determine s_i corresponding to a given point ρ on L_m we substitute $\rho - A - \sum_{i=1}^{m} s_i\alpha^i$ for α_i in Δ_{α^m}. We have then

$$0 = \begin{pmatrix} \alpha^i \\ \rho - A \end{pmatrix} \Delta_{\alpha^m} - s_i\,\Delta_{\alpha^m} \qquad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

where the symbol $\begin{pmatrix} \alpha^i \\ \rho - A \end{pmatrix} \Delta_{\alpha^m}$ denotes the scalar determinant (2.1) in which α^i in the first row is replaced by $(\rho - A)$.

We now introduce a system of m vectors reciprocal to α^i. Remembering the relations

$$\begin{Bmatrix}\alpha^k\\ \alpha^l\end{Bmatrix}\begin{Bmatrix}\alpha^k\\ \alpha^h\end{Bmatrix}_c \begin{matrix} = \Delta_{\alpha^m} \text{ if } l = h \\ = 0 \quad\quad \text{if } l \neq h\end{matrix} \qquad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

where $\begin{Bmatrix}\alpha^k\\ \alpha^h\end{Bmatrix}_c$ denotes the minor of $\begin{Bmatrix}\alpha^k\\ \alpha^h\end{Bmatrix}$ in Δ_{α^m} preceded by the proper sign, we define the system of m reciprocal vectors $\alpha^{(h)}$ by means of the equations

$$\alpha^{(h)} = \alpha^k \begin{Bmatrix}\alpha^k\\ \alpha^h\end{Bmatrix}_c \Big/ \Delta_{\alpha^m} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

so that

$$\begin{Bmatrix}\alpha^{(h)}\\ \alpha^l\end{Bmatrix} = \delta^h_l.$$

Introducing the reciprocal vectors (2.5) may be written as

$$\begin{Bmatrix}\rho - A\\ \alpha^{(j)}\end{Bmatrix} = \sum_{i=1}^{m} s_i \begin{Bmatrix}\alpha^i\\ \alpha^{(j)}\end{Bmatrix} = s_j. \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

§ 3. Let $\beta^j = g^j(x)\sqrt{dx}$ be a vector not belonging to the subspace L_m. Then we can form a new vector

$$\bar{\alpha}^j = \sum_{i=1}^{m} \alpha_i \begin{pmatrix}\alpha^i\\ \beta^j\end{pmatrix} \Delta_{\alpha^m}/\Delta_{\alpha^m} \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

which lies on L_m and is the projection of β^j on L_m.
The vector $\beta'^j = \beta^j - \bar{\alpha}^j$ is normal to L_m, for taking any vector α^k belonging to L_m let us form the equation

$$\begin{Bmatrix}\beta'^j\\ \alpha^k\end{Bmatrix} = \begin{Bmatrix}\beta^j\\ \alpha^k\end{Bmatrix} - \begin{Bmatrix}\bar{\alpha}^j\\ \alpha^k\end{Bmatrix}.$$

Applying (3.1) we have then

$$\Delta_{\alpha^m}\begin{Bmatrix}\beta'^j\\ \alpha^k\end{Bmatrix} = \begin{Bmatrix}\beta^j\\ \alpha^k\end{Bmatrix}\Delta_{\alpha^m} - \sum_{i=1}^{m}\begin{Bmatrix}\alpha^i\\ \alpha^k\end{Bmatrix}\begin{pmatrix}\alpha^i\\ \beta^j\end{pmatrix}\Delta_{\alpha^m} = \begin{Bmatrix}\beta^j & \alpha^1 & \alpha^2 & \ldots & \alpha^m\\ \alpha^k & \alpha^1 & \alpha^2 & \ldots & \alpha^m\end{Bmatrix} = 0. \quad (3.2)$$

Since

$$\begin{Bmatrix}\bar{\alpha}^j\\ \beta'^j\end{Bmatrix} = 0 \text{ we have } \begin{Bmatrix}\beta'^j\\ \beta'^j\end{Bmatrix} = \begin{Bmatrix}\beta^j\\ \beta'^j\end{Bmatrix}.$$

Let ν denote the perpendicular on L_m from a given point B.
Then

$$\nu = (A-B) - \sum_{i=1}^{m}\alpha^i\begin{pmatrix}\alpha^i\\ A-B\end{pmatrix}\Delta_{\alpha^m}/\Delta_{\alpha^m}$$

and

$$\begin{Bmatrix}\nu\\ \nu\end{Bmatrix}\Delta_{\alpha^m} = \begin{Bmatrix}\nu\\ A-B\end{Bmatrix}\Delta_{\alpha^m} = \begin{Bmatrix}A-B & \alpha^1 & \alpha^2 & \ldots & \alpha^m\\ A-B & \alpha^1 & \alpha^2 & \ldots & \alpha^m\end{Bmatrix} \quad \ldots \quad \ldots \quad (3.3)$$

gives the length of the perpendicular ν.

We proceed now to replace the m vectors α^i of L_m by a system of m mutually orthogonal vectors α^{*i}.

Let us start with the vector α^1 at the point A. Taking α^2 we replace it by

$$\alpha^{*2} = \alpha^2 - \alpha^1 \begin{Bmatrix} \alpha^2 \\ \alpha^1 \end{Bmatrix} \Big/ \begin{Bmatrix} \alpha^1 \\ \alpha^1 \end{Bmatrix} \quad \dots \quad \dots \quad \dots \quad (3.4)$$

which is orthogonal to α^1 and lies in the plane of α^1, α^2.
Next, α^3 is replaced by

$$\alpha^{*3} = \alpha^3 - \sum_{i=1}^{2} \alpha^i \begin{pmatrix} \alpha^i \\ \alpha^3 \end{pmatrix} \Delta_{\alpha^2} / \Delta_{\alpha^2} \quad \dots \quad \dots \quad \dots \quad (3.5)$$

which is orthogonal to the plane of α^1, α^2 and lies on the three-dimensional space formed by α^1, α^2, α^3. Proceeding thus we finally replace α^m by

$$\alpha^{*m} = \alpha^m - \sum_{i=1}^{m-1} \alpha^i \begin{pmatrix} \alpha^i \\ \alpha^m \end{pmatrix} \Delta_{\alpha^{m-1}} / \Delta_{\alpha^{m-1}} \quad \dots \quad \dots \quad (3.6)$$

which is orthogonal to the $(m-1)$-dimensional subspace formed by the vectors $\alpha^1, \alpha^2 \dots \alpha^{m-1}$.

From (3.6) we have

$$\Delta_{\alpha^{m-1}} \begin{Bmatrix} \alpha^{*m} \\ \alpha^m \end{Bmatrix} = \Delta_{\alpha^{m-1}} \begin{Bmatrix} \alpha^{*m} \\ \alpha^{*m} \end{Bmatrix} = \Delta_{\alpha^m}.$$

Hence Δ_{α^m} may be expressed in the form

$$\Delta_{\alpha^m} = \begin{Bmatrix} \alpha^1 \\ \alpha^1 \end{Bmatrix} \begin{Bmatrix} \alpha^{*2} \\ \alpha^{*2} \end{Bmatrix} \cdots \begin{Bmatrix} \alpha^{*m} \\ \alpha^{*m} \end{Bmatrix}. \quad \dots \quad \dots \quad (3.7)$$

§ 4. Consider a pair of subspaces L_m and M_l given by the equations

$$\rho = A + \sum_{i=1}^{m} s_i \alpha^i$$

$$\rho = B + \sum_{j=1}^{l} t_j \beta^j \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.1)$$

of m and l dimensions respectively ($m \geqslant l$). If the vectors α^i, β^j be all linearly independent of one another, we can form the scalar determinant

$$\Delta_{\alpha^m \beta^l} = \begin{Bmatrix} \alpha^1 & \alpha^2 & \dots & \alpha^m & \beta^1 & \dots & \beta^l \\ \alpha^1 & \alpha^2 & \dots & \alpha^m & \beta^1 & \dots & \beta^l \end{Bmatrix}. \quad \dots \quad \dots \quad \dots \quad (4.2)$$

of $(m+l)$th order. If, however, some of the vectors β^j be parallel to L_m, (4.2) will be reduced to

$$\Delta_{\alpha^m \beta^k} = \begin{Bmatrix} \alpha^1 & \alpha^2 & \dots & \alpha^m & \beta^1 & \dots & \beta^k \\ \alpha^1 & \alpha^2 & \dots & \alpha^m & \beta^1 & \dots & \beta^k \end{Bmatrix} \quad \dots \quad \dots \quad (4.3)$$

if the $(l-k)$ mutually independent vectors $\beta^{k+1}, \beta^{k+2} \dots \beta^l$ be linearly connected with α^i.

Replacing β^j's in (4.2) and (4.3) by β'^j's defined by

$$\beta'^j = \beta^j - \sum_{i=1}^{m} \alpha^{(i)} \begin{Bmatrix} \alpha^i \\ \beta^j \end{Bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

and noting $\begin{Bmatrix} \beta'^j \\ \alpha^k \end{Bmatrix} = 0$, $(k = 1, 2, \ldots m)$, one can see that the vectors will form a linear subspace orthogonal to L_m of l and k dimensions respectively.

Let γ^j be a vector not lying in either L_m or M_l. Then the vectors

$$\gamma^j_{(\alpha)} = \gamma^j - \sum_{i=1}^{m} \alpha^i \begin{pmatrix} \alpha^i \\ \gamma^j \end{pmatrix} \Delta_{\alpha^m} / \Delta_{\alpha^m} \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

and

$$\gamma^j_{(\beta)} = \gamma^j - \sum_{k=1}^{l} \beta^k \begin{pmatrix} \beta^k \\ \gamma^j \end{pmatrix} \Delta_{\beta^l} / \Delta_{\beta^l}$$

are respectively orthogonal to L_m and M_l.

From the above we get

$$\Delta_{\alpha^m} \begin{Bmatrix} \gamma^j_{(\alpha)} \\ \gamma^j_{(\beta)} \end{Bmatrix} = \begin{Bmatrix} \gamma^j & \alpha^1 \; \alpha^2 \ldots \alpha^m \\ \gamma^j_{(\beta)} & \alpha^1 \; \alpha^2 \ldots \alpha^m \end{Bmatrix}$$

$$\Delta_{\beta^l} \begin{Bmatrix} \gamma^j_{(\beta)} \\ \gamma^j_{(\alpha)} \end{Bmatrix} = \begin{Bmatrix} \gamma^j & \beta^1 \; \beta^2 \ldots \beta^l \\ \gamma^j_{(\alpha)} & \beta^1 \; \beta^2 \ldots \beta^l \end{Bmatrix}. \quad \ldots \quad \ldots \quad (4.6)$$

The above relations show that if $\gamma^j_{(\beta)}$ is parallel to L_m then $\gamma^j_{(\alpha)}$ is parallel to M_l and the two normals are mutually orthogonal.

Let us now find the common normal to the subspaces L_m and M_l when the compound dominant $\Delta_{\alpha^m \beta^l}$ is of $(m+l)$th order.

The common normal ν is defined by the vector

$$\nu = A - B + \sum_{i=1}^{m} s_i \alpha^i - \sum_{j=1}^{l} t_j \beta^j \quad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

satisfying the relations

$$\begin{pmatrix} \alpha^i \\ \nu \end{pmatrix} \Delta_{\alpha^m \beta^l} = \begin{pmatrix} \beta^j \\ \nu \end{pmatrix} \Delta_{\alpha^m \beta^k} = 0 \begin{pmatrix} i = 1, 2 \ldots m \\ j = 1, 2, \ldots l \end{pmatrix}.$$

To determine the parameters s_i and t_j corresponding to ν, we have the equations

$$\begin{pmatrix} \alpha^i \\ A-B \end{pmatrix} \Delta_{\alpha^m \beta^l} + s_i \, \Delta_{\alpha^m \beta^l}$$

$$\begin{pmatrix} \beta^j \\ A-B \end{pmatrix} \Delta_{\alpha^m \beta^l} - t_j \, \Delta_{\alpha^m \beta^l}. \quad \ldots \quad \ldots \quad \ldots \quad (4.8)$$

The square of the length of the common normal is given by

$$\Delta_{\alpha^m \beta^l}\begin{Bmatrix}\nu\\ \nu\end{Bmatrix} = \begin{Bmatrix}A-B\ \alpha^1 & \alpha^2 \ldots \alpha^m\ \beta^1 \ldots \beta^l\\ A-B\ \alpha^1 & \alpha^2 \ldots \alpha^m\ \beta^1 \ldots \beta^l\end{Bmatrix}. \qquad (4.9)$$

If the compound dominant is of the reduced form

$$\Delta_{\alpha^m \beta^k} = \begin{Bmatrix}\alpha^1 \ldots \alpha^m\ \beta^1 \ldots \beta^k\\ \alpha^1 \ldots \alpha^m\ \beta^1 \ldots \beta^k\end{Bmatrix} \qquad (4.10)$$

the common normal ν will satisfy the relations

$$\begin{pmatrix}\alpha^i\\ \nu\end{pmatrix}\Delta_{\alpha^m \beta^k} = \begin{pmatrix}\beta^j\\ \nu\end{pmatrix}\Delta_{\alpha^m \beta^k} = 0\begin{pmatrix}i = 1, 2 \ldots m\\ j = 1, 2, \ldots k\end{pmatrix}.$$

Since $\beta^{k+1}, \ldots \beta^l$ are all linearly connected with the vectors α^i, we can write each of them as

$$\alpha^i\begin{Bmatrix}\alpha^{(i)}\\ \beta^s\end{Bmatrix}.$$

Making use of the dominant (4.10), we have then

$$0 = \begin{pmatrix}\alpha^i\\ A-B\end{pmatrix}\Delta_{\alpha^m \beta^k} + s_i\,\Delta_{\alpha^m \beta^k} - \sum_{j=k+l}^{l} t_i\begin{Bmatrix}\alpha^{(i)}\\ \beta^j\end{Bmatrix}\Delta_{\alpha^m \beta^k}$$

$$0 = \begin{pmatrix}\beta^j\\ A-B\end{pmatrix}\Delta_{\alpha^m \beta^k} - t_j\,\Delta_{\alpha^m \beta^k}\ (j = 1, 2, \ldots k. \qquad (4.11)$$

Now ν in (4.7) is expressible as

$$\nu = A-B+\sum_{i=1}^{m}\alpha^i\left[s_i - \sum_{j=k+1}^{l} t_j\begin{Bmatrix}\alpha^{(i)}\\ \beta^j\end{Bmatrix}\right] - \sum_{j=1}^{k} t_j\,\beta^j. \qquad (4.12)$$

Using (4.11) ν is given by

$$\nu = A-B-\sum_{i=1}^{m}\alpha^i\begin{pmatrix}\alpha^i\\ A-B\end{pmatrix}\Delta_{\alpha^m \beta^k}/\Delta_{\alpha^m \beta^k} - \sum_{j=1}^{k}\beta^j\begin{pmatrix}\beta^j\\ A-B\end{pmatrix}\Delta_{\alpha^m \beta^k}/\Delta_{\alpha^m \beta^k}. \qquad (4.13)$$

To find the angle θ between the two linear subspaces L_m and M_l, we project the subspace

$$\rho = \sum_{j=1}^{l} t_j\,\beta^j \qquad (4.14)$$

upon the subspace

$$\rho = \sum_{i=1}^{m} s_i\,\alpha^i \qquad (4.15)$$

both passing through the origin. Let us represent the projected subspace by

$$\rho = \sum_{j=1}^{l}\bar{s}_j\,\bar{\alpha}^j. \qquad (4.16)$$

Now the angle θ being equal to the angle between (4.14) and (4.16) we have

$$\cos^2\theta = \begin{Bmatrix} \bar{\alpha}^1 & \bar{\alpha}^2 & \dots & \bar{\alpha}^l \\ \bar{\alpha}^1 & \bar{\alpha}^2 & \dots & \bar{\alpha}^l \end{Bmatrix} \Big/ \begin{Bmatrix} \beta^1 & \beta^2 & \dots & \beta^l \\ \beta^1 & \beta^2 & \dots & \beta^l \end{Bmatrix}. \quad \dots \quad \dots (4.17)$$

It can be shown (Ghosh 1936*b*) that

$$\begin{Bmatrix} \bar{\alpha}^1 & \bar{\alpha}^2 & \dots & \bar{\alpha}^l \\ \bar{\alpha}^1 & \bar{\alpha}^2 & \dots & \bar{\alpha}^l \end{Bmatrix} = \frac{1}{\Delta_{\alpha^m}} \sum \begin{pmatrix} \alpha^{i_1} & \alpha^{i_2} & \dots & \alpha^{i_l} \\ \beta^1 & \beta^2 & \dots & \beta^l \end{pmatrix} \Delta_{\alpha^m} \begin{Bmatrix} \alpha^{i_1} & \alpha^{i_2} & \dots & \alpha^{i_l} \\ \beta^1 & \beta^2 & \dots & \beta^l \end{Bmatrix} \quad (4.18)$$

where the summation extends over all l-combinations $(i_1, i_2 \ldots i_l)$ of the integers 1, 2. . . . . m arranged in natural order.

In developing the geometry of function space in connection with some physical applications Synge (1957) gave a lucid treatment of linear subspaces in function space. The present determinantal method is, however, more comprehensive. Besides a neat presentation of results, the general theorems can all be discussed in a more direct and elegant manner.

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