INFINITESIMAL GEOMETRY IN HILBERT SPACE—II

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From a set of elementary vectors of the first order involving a single variable we show how an infinite variety of infinitesimal vectors of higher order can be framed. Different sets of similar vectors generate linear subspaces which are, in general, distinct from one another, but the general treatment of such subspaces does not differ much in individual cases. A scheme of extending the above to include vectors involving any number of variables is also discussed.

§ 1. In the first part (Ghosh 1969) we have dealt with the fundamentals of the infinitesimal geometry in Hilbert space and considered some general properties of linear subspaces formed by a set of vectors of the first order involving a single variable. The object of the present paper is to introduce infinitesimal higher order vectors of pure and mixed types involving several variables. Choosing sets of similar vectors among them one can form linear subspaces, mutually exclusive, but possessing similar properties.

§ 2. Consider a pair of elementary vectors α^1, α^2 having components along the mutually orthogonal coordinate axes $\sqrt{dx_q}$ given by

$$\alpha^1_q = f^1(x_q)\sqrt{dx_q}, \quad \alpha^2_q = f^2(x_q)\sqrt{dx_q} \quad (q = 1, 2, \ldots \infty) \qquad .. \quad (2.1)$$

They have finite scalar product

$$\begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix} = \int_a^b f^1(x) f^2(x)\, dx \qquad .. \quad .. \quad .. \quad (2.2)$$

the square of the norm being respectively

$$\begin{Bmatrix} \alpha^1 \\ \alpha^1 \end{Bmatrix} = \int_a^b \{f^1(x)\}^2\, dx, \quad \begin{Bmatrix} \alpha^2 \\ \alpha^2 \end{Bmatrix} = \int_a^b \{f^2(x)\}^2\, dx \qquad .. \quad .. \quad (2.3)$$

Let us write the components of the pair of vectors in the form of an array

$$\begin{vmatrix} \alpha^1_1 & \alpha^1_2 \ldots \alpha^1_q \ldots \infty \\ \alpha^2_1 & \alpha^2_2 \ldots \alpha^2_q \ldots \infty \end{vmatrix} \qquad .. \quad .. \quad .. \quad (2.4)$$

and construct a two-fold vector α^{12} with components

$$\alpha^{12}_{(qr)} = \begin{vmatrix} \alpha^1_q & \alpha^1_r \\ \alpha^2_q & \alpha^2_r \end{vmatrix} \quad (q \neq r = 1, 2, \ldots \infty) \qquad .. \quad .. \quad (2.5)$$

Remembering that the square of the array (2.4) is equal to the square of the norm of the vector α^{12} we get

$$\begin{Bmatrix} \alpha^{12} \\ \alpha^{12} \end{Bmatrix} = \sum_{q,r} \begin{vmatrix} \alpha^1_q & \alpha^1_r \\ \alpha^2_q & \alpha^2_r \end{vmatrix}^2 = \begin{vmatrix} \begin{Bmatrix} \alpha^1 \\ \alpha^1 \end{Bmatrix} & \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix} \\ \begin{Bmatrix} \alpha^2 \\ \alpha^1 \end{Bmatrix} & \begin{Bmatrix} \alpha^2 \\ \alpha^2 \end{Bmatrix} \end{vmatrix} = \begin{Bmatrix} \alpha^1 & \alpha^2 \\ \alpha^1 & \alpha^2 \end{Bmatrix} \quad \ldots \quad \ldots \quad (2.6)$$

It is to be noted that in the structure of the vector α^{12} infinitesimals occur in groups of two.

If β^{12} denote the two-fold vector constructed from another pair β^1, β^2, then

$$\begin{Bmatrix} \alpha^{12} \\ \beta^{12} \end{Bmatrix} = \sum_{q,r} \begin{vmatrix} \alpha^1_q & \alpha^1_r \\ \alpha^2_q & \alpha^2_r \end{vmatrix} \begin{vmatrix} \beta^1_b & \beta^1_r \\ \beta^2_q & \beta^2_r \end{vmatrix} = \begin{Bmatrix} \alpha^1 & \alpha^2 \\ \beta^1 & \beta^2 \end{Bmatrix} \quad \ldots \quad \ldots \quad (2.7)$$

Next we consider a three-fold vector α^{123} generated by three elementary vectors α^1, α^2, α^3. The components of α^{123} are given by

$$\alpha^{123}_{(qrs)} = \begin{vmatrix} \alpha^1_q & \alpha^1_r & \alpha^1_s \\ \alpha^2_q & \alpha^2_r & \alpha^2_s \\ \alpha^3_q & \alpha^3_r & \alpha^3_s \end{vmatrix} \quad (q \neq r \neq s = 1, 2, \ldots \infty) \quad \ldots \quad \ldots \quad (2.8)$$

In a three-fold vector infinitesimals occur in groups of three. The following relations are obtained easily:

$$\begin{Bmatrix} \alpha^{123} \\ \alpha^{123} \end{Bmatrix} = \begin{Bmatrix} \alpha^1 & \alpha^2 & \alpha^3 \\ \alpha^1 & \alpha^2 & \alpha^3 \end{Bmatrix} \cdot \begin{Bmatrix} \alpha^{123} \\ \beta^{123} \end{Bmatrix} = \begin{Bmatrix} \alpha^1 & \alpha^2 & \alpha^3 \\ \beta^1 & \beta^2 & \beta^3 \end{Bmatrix} \quad \ldots \quad \ldots \quad (2.9)$$

With k elementary vectors $\alpha^1, \alpha^2, \ldots \alpha^k$ we can form a k-fold vector $\alpha^{12 \cdots k}$, whose components in determinantal form are given by

$$\alpha^{12 \ldots k}_{(r_1 r_2 \ldots r_k)} = \begin{vmatrix} \alpha^1_{r_1} & \alpha^1_{r_2} & \ldots & \alpha^1_{r_k} \\ \alpha^2_{r_1} & \alpha^2_{r_2} & \ldots & \alpha^2_{r_k} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha^k_{r_1} & \alpha^k_{r_2} & \ldots & \alpha^k_{r_k} \end{vmatrix} \quad (r_1 \neq r_2 \neq \ldots \neq r_k = 1, 2, \ldots \infty) \quad \ldots \quad (2.10)$$

The square of the norm of the vector $\alpha^{12 \cdots k}$ is

$$\begin{Bmatrix} \alpha^1 & \alpha^2 & \ldots & \alpha_k \\ \alpha^1 & \alpha^2 & \ldots & \alpha_k \end{Bmatrix}$$

and the scalar product of the vector with another similar vector $\beta^{12 \cdots k}$ is

$$\begin{Bmatrix} \alpha^1 & \alpha^2 & \ldots & \alpha_k \\ \beta^1 & \beta^2 & \ldots & \beta_k \end{Bmatrix}$$

In a k-fold vector infinitesimals occur in groups of k. By introducing these k-fold vectors one can notice that an extension of Riemann integration applicable to a set of k functions has been made.

The above are instances of infinitesimal pure vectors of higher order.

§ 3. To illustrate mixed types of vector we shall consider (i) three two-fold vectors generating a 3×2-fold vector and (ii) two three-fold vectors generating a 2×3-fold vector.

(i) Let the three two-fold vectors be denoted by α^{12}, α^{34}, α^{56}. With three sets of components let us form the array

$$\begin{vmatrix} \alpha^{12}_{(12)} & \alpha^{12}_{(13)} & \dots & \alpha^{12}_{(pq)} & \dots & \infty \\ \alpha^{34}_{(12)} & \alpha^{34}_{(13)} & \dots & \alpha^{34}_{(rs)} & \dots & \infty \\ \alpha^{56}_{(12)} & \alpha^{56}_{(13)} & \dots & \alpha^{56}_{(tu)} & \dots & \infty \end{vmatrix} \quad (p \neq q,\ r \neq s,\ t \neq u) \qquad \dots \quad (3.1)$$

Now the square of the array (3.1) is given by the determinant

$$\begin{vmatrix} \begin{Bmatrix} \alpha^{12} \\ \alpha^{12} \end{Bmatrix} & \begin{Bmatrix} \alpha^{12} \\ \alpha^{34} \end{Bmatrix} & \begin{Bmatrix} \alpha^{12} \\ \alpha^{56} \end{Bmatrix} \\ \begin{Bmatrix} \alpha^{34} \\ \alpha^{12} \end{Bmatrix} & \begin{Bmatrix} \alpha^{34} \\ \alpha^{34} \end{Bmatrix} & \begin{Bmatrix} \alpha^{34} \\ \alpha^{56} \end{Bmatrix} \\ \begin{Bmatrix} \alpha^{56} \\ \alpha^{12} \end{Bmatrix} & \begin{Bmatrix} \alpha^{56} \\ \alpha^{34} \end{Bmatrix} & \begin{Bmatrix} \alpha^{56} \\ \alpha^{56} \end{Bmatrix} \end{vmatrix} = \begin{Bmatrix} \alpha^{12} & \alpha^{34} & \alpha^{56} \\ \alpha^{12} & \alpha^{34} & \alpha^{56} \end{Bmatrix} \qquad \dots \quad \dots \quad (3.2)$$

The above is to be evaluated by referring to (2.6) and (2.7). Denoting the 3×2-fold vector by $\alpha^{12,\,34,\,56}$ we now write the components in the form

$$\alpha^{12,\,34,\,56}_{(pq)(rs)(tu)} = \begin{vmatrix} \alpha^{12}_{(pq)} & \alpha^{12}_{(rs)} & \alpha^{12}_{(tu)} \\ \alpha^{34}_{(pq)} & \alpha^{34}_{(rs)} & \alpha^{34}_{(tu)} \\ \alpha^{56}_{(pq)} & \alpha^{56}_{(rs)} & \alpha^{56}_{(tu)} \end{vmatrix} \quad (pq \neq rs \neq tu) \qquad \dots \quad \dots \quad (3.3)$$

The square of the norm of this vector is given by (3.2).

(ii) Let the two three-fold vectors be denoted by α^{123}, α^{456}. With the two sets of components let us form the array

$$\begin{vmatrix} \alpha^{123}_{(123)} & \alpha^{123}_{(124)} & \dots & \alpha^{123}_{(pqr)} & \dots & \infty \\ \alpha^{456}_{(123)} & \alpha^{456}_{(124)} & \dots & \alpha^{456}_{(stu)} & \dots & \infty \end{vmatrix} \qquad \dots \quad \dots \quad \dots \quad (3.4)$$

The square of the array (3.4) is obtained as

$$\begin{vmatrix} \begin{Bmatrix} \alpha^{123} \\ \alpha^{123} \end{Bmatrix} & \begin{Bmatrix} \alpha^{123} \\ \alpha^{456} \end{Bmatrix} \\ \begin{Bmatrix} \alpha^{456} \\ \alpha^{123} \end{Bmatrix} & \begin{Bmatrix} \alpha^{456} \\ \alpha^{456} \end{Bmatrix} \end{vmatrix} = \begin{Bmatrix} \alpha^{123} & \alpha^{456} \\ \alpha^{123} & \alpha^{456} \end{Bmatrix} \qquad \dots \quad \dots \quad \dots \quad (3.5)$$

which can be further reduced by applying (2.9).

Denoting the 2×3-fold vector by $\alpha^{123,\,456}$ we can write the components in the form

$$\alpha^{123,\,456}_{(pqr)(stu)} = \begin{vmatrix} \alpha^{123}_{(pqr)} & \alpha^{123}_{(stu)} \\ \alpha^{456}_{(pqr)} & \alpha^{456}_{(stu)} \end{vmatrix} \quad (pqr) \neq (stu) \qquad \ldots \qquad \ldots \quad (3.6)$$

The square of the norm of the vector has the value (3.5). We can thus obtain any number of infinitesimal vectors of various types having infinite number of components expressed in determinantal form. Addition and scalar product are, however, defined among a set of similar vectors. Scalar product of two vectors $A+B$, $C+D$ is given by the usual formula.

A system of m 'similar' vectors $\hat{\alpha}^i$ of the same type can generate an m-dimensional linear subspace $\hat{L}_m$ given by the equation

$$\hat{\rho} = \hat{A} + \sum_{i=1}^{m} s_i \hat{\alpha}^i, \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (3.7)$$

where (^) indicates that the vectors involved in (3.7) are all similar. For different types of the sets of similar vectors the subspaces represented by (3.7) are mutually exclusive. The general treatment, as adopted in Part I (Ghosh 1969), will not differ much in individual cases. The evaluation of the scalar determinants involved will, of course, depend on the particular choice of the system of similar vectors.

§ 4. Following Riemann process for repeated integral we now introduce infinitesimal vectors of higher rank, such as $f(x,\, y)\sqrt{dx}\sqrt{dy}$ and $f(x,\, y,\, z)\sqrt{dx}\sqrt{dy}\sqrt{dz}$ involving two and three variables respectively; y, z ranging over the intervals $[c,\, d]$ $[e,\, f]$. These may be called repeated vectors of second and third ranks. The square of the norm of the first is given by

$$\int_c^d \int_a^b \{f(x,\, y)\}^2\, dx\, dy \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (4.1)$$

The scalar product of this with a vector $\phi(x)\sqrt{dx}$ is the vector

$$\left[\int_a^b f(x,\, y)\phi(x)\, dx\right]\sqrt{dy}, \qquad \ldots \qquad \ldots \qquad \ldots \quad (4.2)$$

whence assuming $f(x,\, y)$ to be symmetric we can set up an eigenvalue equation of Fredholm's type. Introducing repeated vectors of the second rank in place of the vectors α^i in § 2 and § 3 we can modify the structure of all the derived vectors so as to involve two variables x, y. Similar extension is possible involving greater number of variables if higher rank repeated vectors are used.

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REFERENCE

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