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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF FOURTH ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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Consider the nonlinear difference equation

$$\Delta^2 (r_n \Delta^2 (y_n + p_n y_{n-k})) + f(n, y_{\sigma(n)}) = 0; n \in N(n_0),$$

where $\{r_n\}$ and $\{p_n\}$ are positive real sequences and $\{\sigma(n)\}$ is an increasing sequence of integers, k is a nonnegative integer, $f: N(n_0) \times R \to R$ is a continuous function with $u f(n, u) > 0$ for all $u \neq 0$. The authors obtain necessary and sufficient conditions for the existence of nonoscillatory solutions with a specified asymptotic behaviour. They also obtain sufficient conditions for all solutions to be oscillatory. Examples of their results are also inserted.

Key Words : Oscillatory and Asymptotic Behaviour; Fourth Order Nonlinear Neutral Delay Difference Questions

1. INTRODUCTION

This paper is concerned with a class of nonlinear fourth order difference equations of the form

$$\Delta^2 (r_n \Delta^2 (y_n + p_n y_{n-k})) + f(n, y_{\sigma(n)}) = 0; n \in N(n_0), \quad \text{... (1)}$$

where $N(n_0) = \{n_0, n_0 + 1, n_0 + 2, \ldots\}$, n_0 is a nonnegative integer. Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$ and

(c_1) $\{r_n\}$ is a positive sequence of real numbers for $n \in N(n_0)$ such that

$$\sum_{n=n_0}^{\infty} \frac{n}{r_n} = \infty;$$

(c_2) $\{p_n\}$ is a real sequence such that $0 \le p_n < p < 1$ for all $n \in N(n_0)$;

(c_3) k is a nonnegative integer and $\{\sigma(n)\}$ is a sequence of positive integers with $\lim_{n \to \infty} \sigma(n) = \infty$; and

(c_4) $f: N(n_0) \times R \to R$ is continuous and $f(n, u)$ is nondecreasing in u with $u\, f(n, u) > 0$ for all $u \neq 0$ and all $n \in N(n_0)$, and $f(n, \cdot) \not\equiv 0$ eventually.

By a solution of eq. (1) we mean a real sequence $\{y_n\}$ which is defined for $n \geq \min_{m \geq n_0} \{m - k, \sigma(m)\}$ and satisfies (1) for all $n \in N(n_0)$. A solution of eq. (1) is nonoscillatory if it is eventually positive or eventually negative and oscillatory otherwise. Determining oscillation criteria for difference equations has received a great deal of attention in the last few years, see for example, the monographs by Agarwal[1] Agarwal and Wong[2] and Kocic and Ladas[12]. Compared to ordinary difference equations the study of neutral equations and in particular higher order neutral equations, has received considerably less attention, see, for example [3, 6, 7, 8, 10, 13, 14, 22, 23, 25 & 27] and references cited therein. Most of the results obtained for the higher order equations in these papers are for the case $r_n \equiv 1$ for all $n \in N(n_0)$. Therefore our purpose here is to establish some necessary and sufficient conditions for the existence of nonoscillatory solutions of eq. (1) that exhibit certain type of asymptotic behaviour. In addition, we obtain some sufficient conditions for eq. (1) to be oscillatory. Results here are motivated by some results obtained in [5, 9, 11, 15-21, 24, 26 & 28]. Examples, are included to illustrate our results.

2. Some Preparatory Results

In this section, we state and prove some lemmas, which are useful in establishing the main results. For the sake of convenience we will use the following notations.

$$R(n) = \sum_{s=n_0}^{n-1} \sum_{t=n_0}^{s-1} \frac{t}{r_t}$$

and

$$R(n, N) = \sum_{s=N}^{n-1} \sum_{s=N}^{s-1} \frac{(t-N)}{r_t}.$$

In terms of $R(n)$, (c_1) is equivalent to saying that $R(\infty) = \infty$.

Let $\{y_n\}$ be a real sequence. We will also define a companion or associated sequence $\{z_n\}$ of it by

$$z_n = y_n + p_n\, y_{n-k},\ n \in N(n_0), \qquad \dots (2)$$

where $\{p_n\}$ and k have been defined above. First we give some relation between the sequences $\{y_n\}$ and $\{z_n\}$.

Lemma 2.1 — Let $\{y_n\}_{n=n_0}^{\infty}$ be a positive sequence and $\{z_n\}$ be a sequence defined by (2).

(i) If $\lim_{n\to\infty} y_n = \infty$, then $\lim_{n\to\infty} z_n = \infty$.

(ii) If $\{z_n\}$ converges to zero then so does $\{y_n\}$.

(iii) If $\lim_{n \to \infty} p_n = p^* \in [0, 1)$ and $\lim_{n \to \infty} z_n = c \neq 0$, then $\lim_{n \to \infty} y_n = \frac{c}{1 + p^*}$.

(iv) If $\lim_{n \to \infty} \frac{p_n R(n-k)}{R(n)} = q^* \in [0, 1)$ and $\lim_{n \to \infty} \frac{z_n}{R(n)} = c \neq 0$, then $\lim_{n \to \infty} \frac{y_n}{R(n)} = \frac{c}{1 + q^*}$.

PROOF : The first assertion is clear since $z_n > y_n$. The second follows from the definition of $\{z_n\}$. For the proof of (iii) and (iv) see [14].

Lemma 2.2 If $\{y_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of eq. (1), then there are only the following two cases for n large enough.

(I) $y_n > 0, z_n > 0, \Delta z_n > 0, r_n \Delta^2 z_n > 0, \Delta (r_n \Delta^2 z_n) > 0$; and

(II) $y_n > 0, z_n > 0, \Delta z_n > 0, r_n \Delta^2 z_n < 0, \Delta (r_n \Delta^2 z_n) > 0$.

Lemma 2.3 — If $N \geq n_0$ then $\lim_{n \to \infty} \frac{R(n, N)}{R(n)} = 1$.

Lemma 2.4 — If $\{y_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of eq. (1), then there exist an integer $N \in N(n_0)$ and a constant $k_1 > 0$ such that

$$\frac{1}{2} \Delta (r_n \Delta^2 z_n) R(n) \leq z_n \leq k_1 R(n), n \geq N.$$

Lemma 2.5 — Let $\{y_n\}_{n=n_0}^{\infty}$ be an eventually positive solution of eq. (1), then there exists an integer $n_1 \in N(n_0)$ such that for any integer $N \geq n_1$ we have

$$z_n \geq \sum_{s=N}^{n-1} R(s, N) f(s, y_{\sigma(s)}), n \in N.$$

The proofs of Lemmas 2.2 to 2.5 can be modeled as that of in [9] and hence the details are omitted.

Lemma 2.6 — If $\{y_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of eq. (1), then there exists an integer $N \in N(n_0)$ such that

$$\Delta z_n \geq \frac{1}{2} \Delta (r_n \Delta^2 z_n) \Delta R(n) \text{ for } n \geq N.$$

Also if $\sigma(n) \leq n$, then

$$\Delta z_{\sigma(n)} \geq \frac{1}{2} \Delta (r_n \Delta^2 z_n) \Delta R(\sigma(n)) \text{ for } n \geq N. \quad \text{... (3)}$$

PROOF : From Lemma 2.2 we have for $n \geq n_1 \in N(n_0)$, $z_n > 0$, $\Delta z_n > 0$ and $\Delta^2 (r_n \Delta^2 z_n) < 0$. Hence

$$\Delta z_n \geq \sum_{s=n_1}^{n-1} \Delta^2 Z_s = \sum_{s=n_1}^{n-1} \frac{1}{r_s} r_s \Delta^2 Z_s$$

$$\geq \sum_{s=n_1}^{n-1} \frac{1}{r_s} \left(\sum_{t=n_1}^{s-1} \Delta (r_t \Delta^2 z_t) \right)$$

$$\geq \Delta (r_n \Delta^2 z_n) \sum_{s=n_1}^{n-1} \frac{s-n_1}{r_s}$$

$$= \Delta (r_n \Delta^2 z_n) \Delta R (n, n_1). \qquad \ldots (4)$$

From Lemma 2.3, we conclude that there exists an integer $N \geq n_1$ such that $\Delta R (n, n_1) \geq \frac{1}{2} \Delta R(n)$ for $n \geq N$,

and hence

$$\Delta z_n \geq \frac{1}{2} \Delta (r_n \Delta^2 z_n) \Delta R(n) \text{ for } n \geq N.$$

Since $\Delta^2 (r_n \Delta^2 z_n) < 0$ and $\sigma(n) \leq n$, we have

$$\Delta z_{\sigma(n)} \geq \frac{1}{2} \Delta (r_n \Delta^2 z_n) \Delta R(\sigma(n)) \text{ for } n \geq N.$$

The proof is now complete.

Lemma 2.7 — If $\{y_n\}$ is an eventually positive solution of eq. (1), then there exists an integer $N \in I\!N (n_0)$ such that

$$(1 - p_n) z_n \leq y_n \leq z_n \text{ for } n \geq N.$$

PROOF : Let $\{y_n\}$ be an eventually positive solution of eq. (1) for $n \geq N$.

Then from the definition of z_n, we have $z_n \geq y_n$ for $n \geq N$. From Lemma 2.2, we have $z_n > 0$ and $\Delta z_n > 0$ for $n \geq N$. Hence $y_n = z_n - p_n y_{n-k} \geq z_n - p_n z_{n-k} \geq (1 - p_n) z_n$ for $n \geq N$. This completes the proof of the Lemma.

3. EXISTENCE OF NONOSCILLATORY SOLUTIONS

In this section, we derive necessary and sufficient conditions for the existence of nonoscillatory solutions of eq. (1) with some specified asymptotic behaviour. Such existence criteria will be based on the fixed-point theorem, which requires the concept of a set of uniformly Cauchy sequences.

Let B_N be the set of all real sequences defined on the ray $\{N, N+1, N+2, \ldots\}$ of integers where any individual sequence is bounded with respect to the usual supremum norm. A subset S of

B_N is uniformly Cauchy if for every $\varepsilon > 0$, there exists an integer $N_1 \geq N$ such that $|y_i - y_j| < \varepsilon$ holds for any $i, j \geq N_1$ and for any $\{y_n\} \in S$. The following is a fixed point theorem in Cheng and Patulat[5].

Lemma 3.1 — Let S be a closed, bounded and convex subset of B_N. Suppose T is a continuous mapping of B_N. such that $T(S)$ is contained in S, and suppose further that $T(S)$ is uniformly Cauchy. Then T has a fixed point in S.

We begin with the following theorem which provides an existence of a nonoscillatory solution $\{y_n\}$ of eq. (1) such that $\lim_{n\to\infty} \frac{z_n}{R(n)} = \alpha \neq 0$.

Theorem 3.2 — *A necessary and sufficient condition for the eq. (1) to have a nonoscillatory solution* $\{y_n\}$ *such that* $\lim_{n\to\infty} \frac{z_n}{R(n)} = \alpha \neq 0$ *is that*

$$\sum_{m=n_0}^{\infty} f(n, c(1 - P_{\sigma(n)}) R(\sigma(n))) < \infty \qquad \text{... (5)}$$

for some $c \neq 0$.

PROOF : *Necessity* — Let $\{y_n\}$ be a nonoscillatory solution of eq. (1) such that $\lim_{n\to\infty} \frac{z_n}{R(n)} = \alpha \neq 0$. We may assume without loss of generality that $\{y_n\}$ is eventually positive. Then there exist an integer $N \in I\!N(n_0)$ and constants $\alpha_1 > 0, \alpha_2 > 0$ such that

$$\alpha_1 R(\sigma(n)) \leq Z_{\sigma(n)} \leq \alpha_2 R(\sigma(n)) \text{ for } n \geq N. \qquad \text{... (6)}$$

In view of (3) and (6), we obtain

$$y_{\sigma(n)} \geq (1 - P_{\sigma(n)})\, \alpha_1 R(\sigma(n)) \text{ for } n \geq N. \qquad \text{... (7)}$$

Since $\Delta(r_n \Delta^2 z_n) > 0$ by Lemma 2.2, on summing eq. (1) we obtain

$$\sum_{n=N}^{\infty} f(n, y_{\sigma(n)}) < \infty. \qquad \text{... (8)}$$

From (7) and (8) we conclude that

$$\sum_{n=N}^{\infty} f(n, \alpha_1 (1 - p_{\sigma(n)}) R(\sigma(n))) < \infty.$$

Sufficiency — Suppose that (5) holds for some $c \neq 0$. We may assume that $c > 0$, since a similar proof holds if $c < 0$. let $d > 0$ be such that $\frac{4d}{1-p} < c$ and choose $n \in I\!N(n_0)$ so large that

$$\sum_{n=N}^{\infty} f(n, c(1-p_{\sigma(n)})R(\sigma(n))) < \frac{(1-p)d}{8},$$

and
$$N_0 = \min\left\{N, N-k, \inf_{n \geq N} \sigma(n)\right\} \geq n_0.$$

Let B_{N_0} be the linear space of all real sequences $\{y_n\}_{n=N_0}^{\infty}$ such that $\sup_{n \geq N_0} \frac{|y_n|}{R(n)} = \infty$. It is not difficult to see that B_{N_0} is endowed with the norm $\|y\| = \sup_{n \geq N_0} \frac{|y_n|}{R(n)}$ is a Banach space. Consider the following subset

$$S = \{y \in B_{N_0} : 2(1-p)dR(n) \leq y_n \leq 4dR(n) \text{ for } n \geq N \text{ and } y_n = 0 \text{ for } N_0 \leq n < N\}$$

of the Banach space B_{N_0}. Define an operator $F : S \to B_{N_0}$ by

$$(Fy)_n = \begin{cases} (3+p)dR(n) - p_n y_{n-k} + \sum\limits_{s_3=N}^{n-1} \sum\limits_{s_2=N}^{s_3-1} \frac{1}{r_s} \sum\limits_{s_1=N}^{s_2-1} \sum\limits_{s=s_1}^{\infty} f(s, y_{\sigma(s)}), & n \geq N \\ 0 & , N_0 \leq n < N. \end{cases}$$

We assert that the assumptions of Lemma 3.1 are satisfied. First of all, as one can easily verify that S is a closed, bounded and convex subset of B_{N_0}. Next, F is an invariant mapping. Indeed, for any $\{y_n\}$ in S, we have

$$(Fy)_n \leq (3+p)dR(n) + \sum_{s_3=N}^{n-1} \sum_{s_2=N}^{s_3-1} \frac{1}{r_{s_2}} \sum_{s_1=N}^{s_2-1} \frac{(1-p)d}{8}$$

$$\leq (3+p)dR(n) + \frac{(1-p)d}{8} R(n) \leq 4dR(n),$$

and
$$F(y)_n \geq (3+p)dR(n) - p_n y_{n-k} \geq (3+p)dR(n) - 4pdR(n) \geq 2(1-p)dR(n).$$

Next we shall show that F is continuous. Let $\{y^{(i)}\}$ be a sequence in S such that $y^{(i)} \to y = \{y_n\}$. Since S is closed, $y \in S$. Furthermore,

$$|(Fy^{(i)})_n - (Fy)_n| \leq p_n \|y^{(i)} - y\| + \sum_{s_3=N}^{n-1} \sum_{s_2=N}^{s_2-1} \frac{1}{r_{s_2}} \sum_{s_1=N}^{s_2-1} \sum_{s_1=s}^{\infty}$$

$$\left| f\left(s, y^{(i)}_{\sigma(s)}\right) - f(s, y_{\sigma(s)}) \right|$$

$$\leq p \, \| y^{(i)} - y \| + \sum_{s_3=N}^{\infty} \sum_{s_2=N}^{s_2-1} \frac{1}{r_{s_2}} \sum_{s_1=N}^{s_2-1} \sum_{s_1=s}^{\infty} \left| f\left(s, y^{(i)}_{\sigma(s)}\right) - f(s, y_{\sigma(s)}) \right| .$$

Since by continuity

$$\lim_{i \to \infty} \left| f\left(s, y^{(i)}_{\sigma(s)}\right) - f(s, y_{\sigma(s)}) \right| = 0$$

and

$$\left| f\left(s, y^{(i)}_{\sigma(s)}\right) - f(s, y_{\sigma(s)}) \right| < \varepsilon$$

for $n \geq n_0$. We see from Lebesgue's dominated convergence theorem that

$$\lim_{i \to \infty} \left\{ \sup_{n \geq N_0} \left| \frac{(Fy^{(i)})_n}{R(n)} - \frac{(Fy)_n}{R(n)} \right| \right\} = 0,$$

that is $\lim_{i \to \infty} \| (Fy^{(i)})_n - (Fy)_n \| = 0$. Thus F is continuous.

Finally, we prove that FS is uniformly Cauchy. Indeed, let $\{y_n\} \in B_{N_0}$. Then for $m, n \geq N$,

$$| (Fy)_n - (Fy)_m | = | p_n y_{n-k} - p_m y_{m-k} | + \left| \sum_{s_3=n}^{m-1} \sum_{s_2=N}^{s_3-1} \frac{1}{r_{s_2}} \sum_{s_1=N}^{s_2-1} \sum_{s=s_1}^{\infty} f(s, y_{\sigma(s)}) \right|$$

$$\leq | p_n y_{n-k} - p_m y_{m-k} | + \left| \sum_{s_3=n}^{\infty} \sum_{s_2=N}^{s_3-1} \frac{1}{r_{s_2}} \sum_{s_1=N}^{s_2-1} \sum_{s=s_1}^{\infty} f(s, y_{\sigma(s)}) \right|$$

$$\leq 8 \, pd \, R(n) + \frac{(1-p)\, d}{8} R(n).$$

Hence, there exists an integer N_1 such that $\left| \frac{(Fy)_n}{R(n)} - \frac{(Fy)_m}{R(m)} \right| < \varepsilon$ whenever $m, n \geq N_1$. Since ε is independent of $\{y_n\}$, we see that FS is uniformly Cauchy.

By means of Lemma 3.1, there is an element $y^* = \{y^*_n\}$ *in* S such that $y^* = Fy^*$. We may easily verify that y^* is an eventually positive solution of (1). Furthermore from Stolz's[4] theorem

$$\lim_{n \to \infty} \frac{z_n}{R(n)} = \lim_{n \to \infty} \frac{\Delta z_n}{\Delta R(n)} = \lim_{n \to \infty} \frac{r_n \Delta^2 z_n}{r_n \Delta^2 R(n)} = \lim_{n \to \infty} \frac{r_n \Delta^2 z_n}{n}$$

$$= \lim_{n \to \infty} \Delta (r_n \Delta^2 z_n) = (3 + p)\, d.$$

Thus, $\{y_n\}$ is a nonoscillatory solution of eq. (1) with the desired asymptotic property. The proof is now complete.

Theorem 3.3 — *A necessary and sufficient condition for equation (1) to have a nonoscillatory solution $\{y_n\}$ such that* $\lim_{n \to \infty} z_n = \beta \neq 0$ *is that*

$$\sum_{n=n_0}^{\infty} R(n) \,|\, f(n, c(1 - p_{\sigma(n)})) \,| < \infty \qquad \ldots (9)$$

for some $c \neq 0$.

PROOF : *Necessity* — Let $\{y_n\}$ be a nonoscillatory solution of (1) such that $\lim_{n \to \infty} z_n = \beta \neq 0$. We may assume without loss of generality that $\{y_n\}$ is eventually positive. Then there exist an integer $N \in I\!N(n_0)$ and constants $\beta_1 > 0, \beta_2 > 0$ for which $\beta_1 \le z_{\sigma(n)} \le \beta_2$ for $n \ge N$. Hence, from (3), we have

$$y_{\sigma(n)} \ge \beta_1 (1 - p_{\sigma(n)}) \text{ for } n \ge N. \qquad \ldots (10)$$

Multiply eq. (1) by $R(n)$ and summing it from N to $n - 1$, we obtain

$$\sum_{s=N}^{n-1} R(s) f(s, y_{\sigma(s)}) = - \sum_{s=N}^{n-1} R(s)\, \Delta^2 (r_s \Delta^2 z_s)$$

$$= -R(n)\, \Delta (r_n \Delta^2 z_n) + \Delta R(n)\, r_{n+1} \Delta^2 z_{n+1} - (n+1)\, \Delta z_{n+1} + z_{n+3} + \lambda \ldots (11)$$

where λ is a constant. Observe that z_n is subject to the case II of Lemma 2.2, we deduce from (11) that

$$\sum_{n=N}^{\infty} R(n) f(n, y_{\sigma(n)}) < \infty. \qquad \ldots (12)$$

From (10) and (12), we obtain

$$\sum_{m=N}^{\infty} R(n) f(n, \beta_1 (1 - p_{\sigma(n)})) < \infty,$$

and hence the result follows.

Sufficiency — Suppose that (9) holds with $c > 0$. The case of negative c can be treated similarly. Let $d \le \frac{c(1-p)}{2}$ and take $N \in I\!N(n_0)$ so large that

$$\sum_{n=N}^{\infty} R(n) f(n, c(1 - p_{\sigma(n)})) \le \frac{(1-p)}{4} d \,,$$

and $N_0 = \min\left\{N, N-k, \inf_{n \ge N} \sigma(n)\right\} \ge n_0$. Let B_{N_0} be the space considered in the proof of Theorem 3.2 with norm $\|y\| = \sup_{n \ge N} |y_n|$. We define a bounded, closed and convex subset S of B_{N_0} by

$$S = \{y \in B_{N_0} : (1-p)\, d \le y_n \le 2d \ for \ n \ge N \ and \ y_n = y_N \ for \ N_0 \le n < N\}$$

and an operator $F : S \to B_{N_0}$ by

$$(Fy)_n = \begin{cases} (1 + p\, d - p_n y_{n-k} + \sum_{s_3 = N}^{n-1} \sum_{s_2 = s_3}^{\infty} \frac{1}{r_{s_2}} \sum_{s_1 = s_2}^{\infty} \sum_{s = s_1}^{\infty} f(s, y_{\sigma(s)}), & n \ge N \\ (Fy)_N, & N_0 \le n < N \end{cases}$$

Arguing as in Theorem 3.2, we can easily show that F satisfies all conditions of Lemma 3.1. Therefore, there exists $y \in S$ such that $y = Fy$, that is $\{y_n\}$ is a nonoscillatory solution of eq. (1). Since

$$\Delta z_n = \sum_{s_2 = n}^{\infty} \frac{1}{r_{s_2}} \sum_{s_1 = s_2}^{\infty} \sum_{s = s_1}^{\infty} f(s, y_{\sigma(s)}) > 0,$$

it follows that $\lim_{n \to \infty} z_n = \beta \in [(1-p)\, d, 2d)$. This completes the proof of Theorem 3.3.

Now we find conditions under which conditions of Theorem 3.2 (respectively Theorem 3.3) yields

$$\lim_{n \to \infty} \frac{y_n}{R(n)} = \text{constant} \ne 0 \ (\text{respectively} \lim_{n \to \infty} = y_n = \text{constant} \ne 0) \qquad \dots (13)$$

On combining Theorems 3.2 and 3.3 with Lemma 2.1 (iv) and (iii), we have the following result, which ensure the existence of nonoscillatory solution of (1) having property (13).

Theorem 3.4 — *Assume that condition (iv) of Lemma 2.1 holds. Then equation (1) has a nonoscillatory solution* $\{y_n\}$ *such that* $\lim_{n \to \infty} \frac{y_n}{R(n)} = constant \ne 0$, *if and only if (9) is satisfied.*

Theorem 3.5 — *Assume that condition (iii) of Lemma 2.1 holds. Then eq. (1) has a nonoscilatory solution* $\{y_n\}$ *such that* $\lim_{n \to \infty} y_n = constant \ne 0$, *if and only if (a) is satisfied.*

Example 3.1 — Consider the neutral difference equation of the form

$$\Delta^2\left((n+1)(n+2)\, \Delta^2\left(y_n + \frac{n-k}{2(n-k-1)} y_{n-k}\right)\right) + \frac{4(n-l)^3}{n(n+1)(n+2)(n-l-1)^3} y_{n-l}^3 = 0 \qquad \dots (14)$$

where $n \geq \max\{k+l+1, k+2\}$, k and l are positive integers. It is easy to check that condition (9) is satisfied but condition (5) is not satisfied for the eq. (14). Therefore, by Theorems 3.2 and 3.4, there exists no nonoscillatory solution of (14) such that $\lim_{n \to \infty} \frac{y_n}{R(n)}$ = constant $\neq 0$, and by Theorems 3.3 and 3.5, there exists a nonoscillatory solution of (14) such that $\lim_{n \to \infty} \frac{y_n}{R(n)}$ = constant $\neq 0$. In fact $\{y_n\} = \left\{\frac{n-1}{n}\right\}$ is such a solution of eq. (14).

4. Oscillation Theorems

In this section, we establish conditions for the oscillation of all solutions of eq. (1) where f is strongly sublinear or strongly superlinear as given in the following definition.

Definition 4.1 — The function f is called strongly superlinear if there exists a constant $\alpha > 1$ such that for $u \geq v > 0$ or $u \leq v < 0$,

$$\frac{f(n, u)}{|u|^{\alpha} \operatorname{sgn} u} \geq \frac{f(n, v)}{|v|^{\alpha} \operatorname{sgn} v}, \; n \in I\!N(n_0).$$

The function f is called strongly sublinear if there exists a constant β with $0 < \beta < 1$ such that for $u \geq v > 0$ or $u \leq v < 0$,

$$\frac{f(n, u)}{|u|^{\beta} \operatorname{sgn} u} \geq \frac{f(n, v)}{|v|^{\beta} \operatorname{sgn} v}, \; n \in I\!N(n_0).$$

Theorem 4.2 — *Let f be strongly sublinear and $\sigma(n) \leq n$. A necessary and sufficient condition for all solutions of eq. (1) are oscillatory is that*

$$\sum_{n=n_0}^{\infty} |f(n, c(1 - p_{\sigma(n)}) R(n))| = \infty \qquad \text{... (15)}$$

for all $c \neq 0$.

PROOF : The necessity of condition (15) follows from the sufficiency part of Theorem 3.2. Now we prove the sufficiency of condition (15). Assume that there exists a nonoscillatory solution $\{y_n\}$ of eq. (1). Without loss of generality we may assume that $\{y_n\}$ is eventually positive. From Lemmas 2.2, 2.4 and 2.7 there exists $N \geq n_0$ and $k > 0$ such that $z_n > 0, \Delta z_n > 0$ and $\Delta(r_n \Delta^2 z_n) > 0$ for $n \geq N$;

$$y_{\sigma(n)} \geq (1 - p_{\sigma(n)}) z_{\sigma(n)} \text{ for } n \geq N; \qquad \text{... (16)}$$

and

$$\frac{1}{2} \Delta(r_n \Delta^2 z_n) R(n) \leq z_n \leq k R(n); \; n \geq N. \qquad \text{... (17)}$$

Since $\sigma(n) \leq n$ and $\Delta^2(r_n \Delta^2 z_n) < 0$ we have from Lemma 2.4,

$$z_{\sigma(n)} \geq \frac{1}{2} \Delta (r_n \Delta^2 z_n) R(\sigma(n)); n \geq N. \qquad \text{... (18)}$$

From (16)-(18) and strong sublinearity of f, we have

$$\Delta (-(\Delta (r_n \Delta^2 z_n))^{1-\beta}) = (1-\beta) \xi^{-\beta} f(n, y_{\sigma(n)})$$

where $\Delta (r_{n+1} \Delta^2 z_{n+1}) < \xi < \Delta (r_n \Delta^2 z_n)$.

Thus,

$$\Delta (-(\Delta (r_n \Delta^2 z_n))^{1-\beta}) = (1-\beta) \xi^{-\beta} f(n, y_{\sigma(n)})$$

$$\geq (1-\beta) (\Delta (r_n \Delta^2 z_n))^{-\beta} \frac{f(n, y_{\sigma(n)})}{y^{\beta}_{\sigma(n)}} y^{\beta}_{\sigma(n)}$$

$$\geq (1-\beta) (\Delta (r_n \Delta^2 z_n))^{-\beta} \frac{f(n, (1-p_{\sigma(n)}) (z_{\sigma(n)})}{((1-p_{\sigma(n)}) z_{\sigma(n)})^{\beta}}$$

$$\{Z(1-\beta) (\Delta (y_n \Delta^2 Z_n)^{-\beta} f(n,(1-p\sigma_n) \frac{k^{R/\sigma_{(n)}}}{k^{\beta} (R(\sigma(n))^{\beta}} \left(\frac{1}{2} \Delta (y_n \Delta^2 Z_n))^{\beta} R(\sigma(n) \right)$$

$$\geq \frac{(1-\beta)}{(2k)^{\beta}} f(n, (1-p_{\sigma(n)}) kR (\sigma(n))).$$

Summing the last inequality from N to $n - 1$, we obtain

$$\frac{(1-\beta)}{(2k)^{\beta}} \sum_{s=N}^{n-1} f(s, k(1-p_{\sigma(s)}) R(\sigma(s))) \leq (\Delta (r_N \Delta^2 Z_N))^{1-\beta}$$

which leads to

$$\sum_{s=N}^{\infty} f(s, k(1-p_{\sigma(s)}) R(\sigma(s))) < \infty$$

and hence contradicts (15). This completes the proof.

Theorem 4.3 — *Let f be strongly superlinear and $\sigma(n) \geq n$. Then a necessary and sufficient condition for all solutions of eq. (1) are oscillatory is that*

$$\sum_{n=n_0}^{\infty} R(n) \mid f(n, c(1-p_{\sigma(n)})) \mid = \infty \qquad \text{... (19)}$$

for all $c \neq 0$.

PROOF : The necessity of condition (19) follows from the sufficiency part of Theorem 3.3. We prove the sufficiency of condition (19). Assume that there exists a nonoscillatory solution $\{y_n\}$ of eq. (1). Without loss of generality we may assume that $\{y_n\}$ is eventually positive. From Lemmas 2.2, 2.5 and 2.7 there exists $N_1 \geq n_0$ such that

$$z_n > 0, \Delta z_n > 0, \Delta (r_n \Delta^2 z_n) > 0, n \geq N_1;$$

$$y_{\sigma(n)} \geq (1 - p_{\sigma(n)}) z_{\sigma(n)},$$

and
$$z_n \geq \sum_{s=N}^{n-1} R(s, N) f(s, y_{\sigma(s)}), n \geq N \geq N_1. \quad \ldots (20)$$

We have from $z_n > 0, \Delta z_n > 0$, there exists a $k_1 > 0$ such that $z_n \geq k_1$ for $n \geq N$ and hence $y_n \geq k_1 (1 - p_n)$ for $n \geq N$. From strong superlinearity, and $\sigma(n) \geq n$, we have

$$f(n, y_{\sigma(n)}) \geq \frac{f(n, k_1 (1 - p_{\sigma(n)}))}{(k_1 (1 - p_{\sigma(n)}))^\alpha} ((1 - p_{\sigma(n)}) z_{\sigma(n)})^\alpha$$

$$\geq k_1^{-\alpha} z_n^\alpha f(n, k_1 (1 - p_{\sigma(n)})). \quad \ldots (21)$$

From (20) and (21) it follows that

$$z_n \geq \sum_{s=N}^{n-1} k_1^{-\alpha} z_n^\alpha R(s, N) f(s, k_1 (1 - p_{\sigma(s)}))$$

and hence
$$\Delta \left(\left[\sum_{s=N}^{n-1} k_1^{-\alpha} z_s^\alpha R(s, N) f(s, k_1 (1 - p_{\sigma(s)})) \right]^{1-\alpha} \right)$$

$$= \frac{(1 - \alpha) k_1^{-\alpha} z_n^\alpha R(n, N) f(n, k_1 (1 - p_{\sigma(n)}))}{\left[\sum_{s=N}^{n-1} k_1^{-\alpha} z_s^\alpha R(s, N) f(s, k_1 (1 - p_{\sigma(s)})) \right]^\alpha}$$

$$\leq (1 - \alpha) z_n^{-\alpha} k_1^{-\alpha} z_n^\alpha R(n, N) f(n, k_1 (1 - p_{\sigma(n)}))$$

$$= (1 - \alpha) k_1^{-\alpha} R(n, N) f(n, k_1 (1 - p_{\sigma(n)}))$$

which implies that

$$R (n, N) f (n, k_1 (1 - p_{\sigma(n)})) \leq \frac{k_1^\alpha}{(1 - \alpha)} \Delta$$

$$\left(\left[\sum_{s=N}^{n-1} k_1^{-\alpha} z_s^\alpha R(s, N) f(s, k_1 (1 - p_{\sigma(s)})) \right]^{1-\alpha} \right)$$

Summing the last inequality from N_2 to n and using $\alpha > 1$ we obtain

$$\sum_{s=N_2}^{n} R(s, N) f(s, k_1 (1 - p_{\sigma(s)})) \leq \frac{k_1^{\alpha}}{(\alpha - 1)}$$

$$\left[\sum_{s=N}^{N_2} k_1^{-\alpha} z_s^{\alpha} R(s, N) f(s, k_1 (1 - p_{\sigma(s)})) \right]^{1-\alpha} < \infty.$$

Hence $\sum_{s=N_2}^{\infty} R(s) f(s, k(1 - p_{\sigma(s)})) < \infty$, a contradiction to (19). The proof is complete.

***Theorem* 4.4** — *Assume that there exists a real sequence $\{q_n\}$ such that*

$$\frac{f(n, u)}{u} \geq M q_n > 0 \text{ for all } u \neq 0, n \geq n_0 \qquad \text{... (22)}$$

and $\sigma(n) = n - l$, where l is a nonnegative integer less than n, for $n \geq n_0$. ... (23)

If thee exists a positive sequence $\{\rho_n\}$ such that

$$\lim_{n \to \infty} \sup \sum_{s=n_0}^{n} \rho_s \left[(1 - p_{s-l}) q_s - \frac{(\Delta \rho_s)^2}{2M \Delta R(s - l) \rho_s^2} \right] = \infty, \qquad \text{... (24)}$$

then all solutions of eq. (1) are oscillatory.

PROOF : Let $\{y_n\}$ b a nonoscillatory solution of (1) and assume without loss of generality that $\{y_n\}$ is eventually positive. From Lemmas 2.2 and 2.7, we have $z_n > 0, z_{n-l} > 0, \Delta z_n > 0$ and $\Delta (r_n \Delta^2 z_n) > 0$ for $n \geq N$ and $y_{n-l} \geq (1 - p_{n-l}) z_{n-l}$.

Define

$$\omega_n = \frac{\rho_n \Delta (r_n \Delta^2 z_n)}{z_{n-l}}; n \geq N.$$

Then in view of Lemma 2.6, (22) and (23) we have

$$\Delta \omega_n \leq \frac{\rho_n \Delta^2 (r_n \Delta^2 z_n) + \Delta (r_n \Delta^2 z_n) \Delta \rho_n}{z_{n-1}} - \frac{\rho_n \Delta (r_n \Delta^2 z_n) \Delta z_{n-1}}{(z_{n-1})^2}$$

$$\leq - M q_n (1 - p_{n-1}) \rho_n + \Delta \rho_n \frac{\omega_n}{\rho_n} - \frac{1}{2\rho_n} \omega_n^2 \Delta R(n - l)$$

$$\leq - M q_n (1 - p_{n-l}) \rho_n \frac{(\Delta \rho_n)^2}{2\rho_n \Delta R(n - l)}.$$

Summing the last inequality from N to $n \geq N$, we obtain

$$\sum_{s=N}^{n} \rho_{(s)} \left[(1 - p_{s-l})\, q_s - \frac{(\Delta \rho_s)^2}{2M\,(\rho_s)^2\, \Delta R(s-l)} \right] \leq \frac{\omega_N}{M}$$

and this contradicts (24). Thus the proof is complete.

For the linear equation

$$\Delta^4 (y_n + p_n y_{n-\tau}) + q_n y_{n-\sigma} = 0, \qquad \text{... (25)}$$

where τ and σ are nonnegative integers less than n, we obtain from Theorem 4.4, the following corollary.

Corollary 4.5 — Suppose $q_n \geq 0$ for all $n \geq n_0$ and there exists a positive sequence $\{\rho_n\}$ such that

$$\lim_{n \to \infty} \sup \sum_{s=n_0}^{n} \rho_s \left[(1 - p_{s-\sigma})\, q_s - \left(\frac{\Delta \rho_s}{s\rho_s} \right)^2 \right] = \infty,$$

then all solutions of eq. (25) are oscillatory.

We conclude this paper with the following examples.

Example 4.1 — Consider the neutral difference equation

$$\Delta^2 \left(n\,(n+1)\, \Delta^2 \left(y_n + \frac{1}{\sqrt{n-1}}\, y_{n-1} \right) \right) + n\, y_{n-1}^{\frac{1}{3}} = 0;\ n \geq 3. \qquad \text{... (26)}$$

It is easy to see that all conditions of Theorem 4.2 are satisfied and hence all solutions of (26) are oscillatory.

Example 4.2 — Theorem 4.3 implies that all solutions of the neutral difference equation

$$\Delta^2 \left(\frac{1}{n+3}\, \Delta^2 \left(y_n + \frac{1}{n+3}\, y_{n-3} \right) \right) + n(n+1)\, y_{n+3}^{3} = 0;\ n \geq 3$$

are oscillatory.

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